

ENTROPY, STABILITY AND HARMONIC MAP FLOW

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ABSTRACT. Inspired by work of Colding-Minicozzi [3] on mean curvature flow, Zhang [16] introduced a notion of entropy stability for harmonic map flow. We build further upon this work in several directions. First we prove the equivalence of entropy stability with a more computationally tractable \mathcal{F} -stability. Then, focusing on the case of spherical targets, we prove a general instability result for high-entropy solitons. Finally, we exploit results of Lin-Wang [11] to observe long time existence and convergence results for maps into certain convex domains and how they relate to generic singularities of harmonic map flow.

1. INTRODUCTION

In [3], Colding and Minicozzi introduced a notion of entropy for mean curvature flow. They define a notion of entropy stability for solitons mean curvature flow and prove its equivalence to a more computationally tractable notion of \mathcal{F} -stability. In this context, a solution to mean curvature flow which only encounters entropy-stable singularities is said to be generic. This is in the sense that one cannot perturb a stable singularity shortly before the singular time to avoid the singularity. Equipped with the equivalence to \mathcal{F} -stability, in [3] it is proved that generic singularities to mean curvature flow are round spheres or cylinders.

Inspired by that work, we aim to understand singularities of the harmonic map heat flow by studying the stability of solitons (see Definition 3.1). Through rescaling arguments ([7, 11]), one can produce soliton limits from singularities of the harmonic map flow, and thus understanding them is central to understanding the long time existence behavior of the flow. First we study a entropy stability condition for the harmonic map heat flow and prove its equivalence to a notion of \mathcal{F} -stability introduced in [16]. The relevant definitions for the statement below are found in §3 and §4.

Theorem 1.1. *Suppose $f : \mathbb{R}^m \rightarrow N$ is a non-cylindrical soliton with polynomial energy density growth. Then f is \mathcal{F} -unstable if and only if there is a compactly supported variation f_s such that $f_0 = f$ and for all $s \neq 0$ one has*

$$\lambda(f_s) < \lambda(f).$$

The notion of \mathcal{F} -stability is a more tractable notion involving the spectrum of a certain kind of second variation operator associated to the the entropy functional (see (3.15)). Given this, one would like to simplify the task of testing for stability further by obtaining a characterization in terms of Rayleigh quotients. The quantity μ_1 in the statement below is the infimum of the relevant Rayleigh quotient (see Definition 5.5), which, due to the noncompactness of the domain, may equal $-\infty$.

Theorem 1.2. *Suppose $f : \mathbb{R}^m \rightarrow N$ is a soliton with polynomial energy density growth. If $\mu_1 < -\frac{3}{2}$ then f is \mathcal{F} -unstable.*

Given the equivalence between λ -stability and \mathcal{F} -stability, we then consider the problem of classifying λ -stable solutions into various target spaces by studying \mathcal{F} -stability via the Rayleigh

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quotient μ_1 . We first note that in principle one should expect a large variety of stable solitons for the harmonic map flow. In particular, as shown in [2], given $m, n \geq 3$, for any homotopically non-trivial class in $C^1(S^m, S^n)$ there is an $\epsilon > 0$ such that any $f_0 : S^m \rightarrow S^n$ in the class with the energy below ϵ is the initial condition for a heat flow which goes singular in finite time, leading to soliton blowup models which should be stable. On the other hand the entropy of these solutions cannot be too small due to the ϵ -regularity results of Struwe [14]. Our result gives a uniform upper bound for the entropy of stable solitons mapping into the sphere, complementing these results.

Theorem 1.3. *Suppose $f : \mathbb{R}^m \rightarrow S^n$ is an entropy stable soliton. Then*

$$\lambda(f) \leq \frac{3n}{4(n-2)}.$$

To prove Theorem 1.3 we first use the equivalence between entropy stability and \mathcal{F} -stability provided by Theorem 1.1. We then construct test variation fields for the \mathcal{F} -stability condition using conformal vector fields on the sphere. We cannot directly show the existence of negative eigenvectors in the space of conformal vector fields and so instead we rely on Theorem 1.2. The use of conformal vector fields in understanding stability of harmonic maps into spheres is classical, see [13].

Lastly, we observe results relating entropy stability and \mathcal{F} -stability to a kind of “dimensional instability” for harmonic map heat flow. Taking a cue from classical results we show in Proposition 5.9 that any nonconstant soliton whose image is contained in a great sphere is entropy unstable. The vector field exhibiting instability is the conformal vector field with poles at the vectors orthogonal to the given great sphere. This suggests that, in studying harmonic maps into spheres, at a singular time one could embed the map into a sphere of one higher dimension as a great sphere, and then perturb the map into the extra dimension to remove the unstable singularity.

As it turns out this strategy is successful, and any map landing strictly in a hemisphere admits a smooth long time solution to harmonic map flow. A well-known result of Gordon [6] says that any harmonic map into a region admitting a convex function is constant. We show a related result which says that if a map has image contained in a sublevel set of a boundary-defining convex function, the solution to harmonic map flow with this initial condition exists smoothly for all time and converges to a point. Lin-Wang show results of this kind, and indeed our result is effectively a corollary of ([11] Theorem 5.4.3). We require a little bit of care, as the results of Lin-Wang implicitly assume completeness of the target, which is not satisfied in our setting. Indeed, we provide examples in §6 that various natural statements about harmonic map flow with convex target are false. These issues are resolved by observing a parabolic maximum principle which shows that sublevel sets of boundary-defining convex functions are preserved under harmonic map flow.

Theorem 1.4. *Let (M^n, g) be a compact Riemannian manifold, and suppose $(N, \partial N, h)$ is a compact Riemannian manifold with boundary which is a sublevel set of a strictly convex boundary-defining function. Given $f : M \rightarrow N$ a smooth map, the solution to harmonic map heat flow with initial condition f exists for all time and converges to a point.*

This theorem in particular implies the result mentioned above that maps into hemispheres admit long time solutions to harmonic map flow which converge to points. This suggests a different way to move past singularities for maps into spheres, by perturbing into a hemisphere of a sphere of one dimension higher. Moreover, this result motivates some conjectures about the structure of solitons mapping into spheres by perturbing the maps into a hemisphere of one dimension higher and studying the behavior of the flow at infinity. This is discussed in §6.3.

Here is an outline of the rest of the paper. In §2 we record some fundamental properties of the \mathcal{F} -functionals and entropy. In §3 refine these results in the case of a Euclidean source. Then in §4 we prove Theorem 1.1. In §5 we give stability and rigidity results for solitons, focusing on the sphere and leading to Theorem 1.3. We conclude in §6 with the proof of Theorem 1.4.

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2. \mathcal{F} -FUNCTIONAL AND ENTROPY

In this section we provide some background and set notation concerning harmonic maps, and provide the basic definitions of the \mathcal{F} -functional and its stability.

2.1. Background and notation. Let (M^m, g) and (N^n, h) be two closed Riemannian manifolds, and $f \in C^\infty(M, N)$. We will consider a connection ∇ defined on the tensor bundle $TM \otimes f^*(TN)$ induced by the Levi-Civita connection on M and N . We will drop all superscripts written on the connections unless necessary, and will generally abuse notation by letting ∇ denote the various connections on different bundles.

When considering local coordinates about some point, we will use Greek indices $\{x^\alpha\}_{\alpha=1}^m$ for coordinates on M and Roman indices $\{y^i\}_{i=1}^n$ for coordinates on N . Furthermore, we will write $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$ and $\partial_i := \frac{\partial}{\partial y^i}$ to span TN . We abuse notation by letting ∂_i simultaneously denote an element of TN and $f^*(TN)$. The associated duals will be given by d^α and d^i , respectively. Let $Tf : TM \rightarrow f^*(TN)$ be the *differential of f* , given in local coordinates by

$$Tf = (\partial_\alpha f^i) d^\alpha \otimes \partial_i.$$

More generally, subscripts will indicate spatial derivatives of various items. The action of ∇ on elements of $f^*(TN)$ has the coordinate form

$$\nabla_\alpha \partial_j = (\partial_\alpha f^i) \nabla_i \partial_j = (\partial_\alpha f^i) \Gamma_{ij}^k \partial_k,$$

where Γ_{ij}^k denotes the connection coefficients of the Levi-Civita connection in local coordinates on N .

2.2. Entropy and solitons. For a smooth map $f \in C^\infty((M, g), (N, h))$, we define the *energy density* by

$$e(f) := \frac{1}{2} |Tf|_{g,h}^2,$$

and the *total energy* by

$$\mathcal{E}(f) := \int_M e(f) dV_g.$$

The *tension field*, which is the negative gradient of the total energy, is given by

$$\tau(f) := \text{tr}_g(\nabla Tf),$$

The *harmonic map heat flow*, or negative gradient flow of the energy, is given by

$$(HMHF) \quad \frac{\partial f_t}{\partial t} = \tau(f_t).$$

We will suppress the dependence on f_t in the notation for τ_t from this point forward. We next define the \mathcal{F} -functional as well as the entropy for a map of Riemannian manifolds.

Definition 2.1. Given (M^m, g) and (N^n, h) Riemannian manifolds, define a functional \mathcal{F} via

$$(2.1) \quad \begin{aligned} \mathcal{F} : C^\infty(M, N) \times \mathbb{R} \times C^\infty(M) &\rightarrow \mathbb{R} \\ \mathcal{F}(f, v, \theta) &= \frac{v}{2} \int_M |Tf|_{g,h}^2 \frac{e^{-\theta}}{(4\pi v)^{m/2}} dV_g. \end{aligned}$$

Moreover we define the *entropy* λ by

$$(2.2) \quad \begin{aligned} \lambda : C^\infty(M, N) &\rightarrow \mathbb{R} \\ \lambda(f) &= \sup_{\{v, \theta : \int_M e^{-\theta} (4\pi v)^{-m/2} dV_g = 1\}} \mathcal{F}(f, v, \theta). \end{aligned}$$

Zhang [16] defines a functional of this kind for maps from \mathbb{R}^m , in direct analogy with Colding-Minicozzi's definition for mean curvature flow. There one only considers weighting against Gaussian densities with different basepoints in $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$. This suffices for the purposes of understanding singularity models in both cases. We have generalized this to a more flexible entropy functional more akin to Perelman's Ricci flow entropy [12], now defined on arbitrary Riemannian manifold by allowing weighting against arbitrary probability measures.

We will adapt Hamilton-Struwe's monotonicity formula [9, 14] to establish a monotonicity for λ in §2.4. These monotone quantities \mathcal{F} and λ are central to understanding singularities of harmonic map flow. Moreover, the critical points of λ are exactly self shrinking solutions. The primary disadvantage of λ is that it does not depend smoothly on $C^\infty(M, N)$. Therefore, to overcome this in using λ to define the stability of solitons, we will demonstrate that it is essentially sufficient to check classical stability of the functional \mathcal{F} . This analysis will be performed in §4 in the case where the source manifold is flat Euclidean space.

2.3. Variational properties. Next we compute the first and second variations of the \mathcal{F} -functional (Definition 2.1). In the next section §3 we will restrict to the case of Euclidean space, and consider a more restricted form of the entropy functional λ . Throughout this section we will suppose that the given geometric data is regular enough so that all integrations by parts are valid. To begin we define a family of flows which will be key in the variations.

First, given a solution to harmonic map heat flow on $[0, t_0)$, and a final value $\Psi_{t_0} : M \rightarrow \mathbb{R}$ we consider the *backwards heat flow* given by

$$(BHF) \quad \begin{cases} \frac{\partial \Psi_t}{\partial t} &= -\Delta \Psi_t, \\ \Psi_t|_{t=t_0} &= \Psi_{t_0}. \end{cases}$$

In the case that Ψ_{t_0} is a Dirac delta function based at x_0 , we say that this is the *backwards heat flow based at (x_0, t_0)* . Next we consider the following two flows: first,

$$(2.3) \quad \begin{cases} \partial_t v_t &= -1, \\ v_t|_{t=0} &= v_0, \end{cases}$$

and then

$$(2.4) \quad \begin{cases} \partial_t \theta_t &= \left(|\nabla \theta_t|^2 - \Delta \theta_t \right) + \frac{m}{2v_t}, \\ \theta_t|_{t=0} &= \theta_0. \end{cases}$$

Lemma 2.2. *Suppose v_t satisfies (2.3) and θ_t satisfies (2.4). Then $\Theta_t := (4\pi v_t)^{-m/2} e^{-\theta_t}$ satisfies (BHF).*

Proof. We simply compute

$$\begin{aligned}
\Delta\Theta_t &= \nabla_\alpha \nabla_\alpha \left[\frac{e^{-\theta_t}}{(4\pi v_t)^{m/2}} \right] \\
&= \nabla_\alpha \left[\frac{-(\nabla_\alpha \theta_t) e^{-\theta_t}}{(4\pi v_t)^{m/2}} \right] \\
&= \frac{-(\Delta\theta_t)}{(4\pi v_t)^{m/2}} + \frac{|\nabla\theta|^2 e^{-\theta_t}}{(4\pi v_t)^{m/2}} \\
&= \frac{e^{-\theta_t}}{(4\pi v_t)^{m/2}} \left(|\nabla\theta_t|^2 - \Delta\theta_t \right) \\
&= \frac{e^{-\theta_t}}{(4\pi v_t)^{m/2}} \left((\partial_t \theta_t) - \frac{m}{2v_t} \right) \\
&= - \left(\frac{4\pi m}{2} \frac{e^{-\theta_t}}{(4\pi v_t)^{m/2+1}} - \frac{(\partial_t \theta_t) e^{-\theta_t}}{(4\pi v_t)^{m/2}} \right) \\
&= -\partial_t \Theta_t.
\end{aligned}$$

The result follows. \square

Definition 2.3. Given $\Theta \in C^\infty(M)$, $\Theta > 0$, let

$$\begin{aligned}
S_\Theta &: C^\infty(M, N) \rightarrow f^*(TN) \\
S_\Theta(f) &= \tau + \left(\frac{\nabla\Theta}{\Theta} \lrcorner Tf \right).
\end{aligned}$$

Proposition 2.4. Given one-parameter families f_s, v_s, θ_s , let

$$\dot{v}_s = \frac{dv_s}{ds}, \quad \dot{f}_s = \frac{df_s}{ds}, \quad \dot{\theta}_s = \frac{d\theta_s}{ds}.$$

Moreover, let $\Theta_s = (4\pi v_s)^{-m/2} e^{-\theta_s}$ as above. Assuming that $f_s \in H_{loc}^1(M, N)$ and f_s satisfies

$$\int_M \left(|\dot{f}_s|^2 + |\nabla f_s|^2 + \left| \frac{\nabla\Theta_s}{\Theta_s} \right|^2 |Tf_s|^2 + |\tau_s|^2 \right) \Theta_s dV_g < +\infty,$$

then

$$(2.5) \quad \frac{d}{ds} [\mathcal{F}(f_s, v_s, \theta_s)] = \frac{1}{2} \int_M \left(\dot{v}_s + v_s \frac{\dot{\Theta}_s}{\Theta_s} \right) |Tf_s|^2 \Theta_s dV_g - v_s \int_M \left\langle \dot{f}_s, S_{\Theta_s}(f_s) \right\rangle \Theta_s dV_g.$$

Proof. We first compute, and apply integration by parts to obtain

$$\begin{aligned}
(2.6) \quad \frac{d}{ds} \left[\int_M |Tf|^2 \Theta dV_g \right] &= 2 \int_M \langle \nabla_s Tf_\alpha, Tf_\alpha \rangle \Theta dV_g + \int_M |Tf|^2 \dot{\Theta} dV_g \\
&= 2 \int_M \langle \nabla \dot{f}, Tf \rangle \Theta dV_g + \int_M |Tf|^2 \dot{\Theta} dV_g.
\end{aligned}$$

We manipulate the first integral,

$$\begin{aligned}
\int_M \langle \nabla \dot{f}, Tf \rangle \Theta dV_g &= \int_M \partial_\alpha \langle \dot{f}, Tf_\alpha \rangle \Theta dV - \int_M \langle \dot{f}, \nabla_\alpha Tf_\alpha \rangle \Theta dV_g \\
&= - \int_M \langle \dot{f}, Tf_\alpha \rangle (\partial_\alpha \Theta) dV_g - \int_M \langle \dot{f}, \nabla_\alpha Tf_\alpha \rangle \Theta dV_g \\
&= - \int_M \left\langle \dot{f}, \left(\frac{\nabla\Theta}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g - \int_M \langle \dot{f}, \tau \rangle \Theta dV_g \\
&= - \int_M \left\langle \dot{f}, S_\Theta(f) \right\rangle \Theta dV_g.
\end{aligned}$$

Incorporating this back into (2.6) we have that

$$\frac{d}{ds} \left[\int_M |Tf_s|^2 \Theta_s dV_g \right] = \int_M \frac{\dot{\Theta}_s}{\Theta_s} |Tf_s|^2 \Theta_s dV_g - 2 \int_M \left\langle \dot{f}_s, S_{\Theta_s}(f_s) \right\rangle \Theta_s dV_g.$$

Therefore

$$\begin{aligned} \frac{d}{ds} [\mathcal{F}(f_s, v_s, \theta_s)] &= \frac{\dot{v}_s}{2} \int_M |Tf_s|^2 \Theta_s dV + \frac{v_s}{2} \frac{d}{ds} \left[\int_M |Tf_s|^2 \Theta_s dV_g \right] \\ &= \frac{1}{2} \int_M \left(\dot{v}_s + v_s \frac{\dot{\Theta}_s}{\Theta_s} \right) |Tf_s|^2 \Theta_s dV_g - v_s \int_M \left\langle \dot{f}_s, S_{\Theta_s}(f_s) \right\rangle \Theta_s dV_g. \end{aligned}$$

The result follows. \square

Before computing the second variation we record a lemma showing how τ varies along a path. The cleanest way to do this is to treat the family as a map on $M \times I$, and then one naturally observes curvature terms arising when commuting time derivatives with connection derivatives. An example of this is in the following lemma.

Lemma 2.5. *Suppose f_s is a one-parameter family of smooth maps for $s \in I \subset \mathbb{R}$. Then*

$$\left(\frac{\partial \tau_s}{\partial s} \right) = \Delta \dot{f}_s + R^N(\dot{f}_s, (Tf_s)_\alpha)(Tf_s)_\alpha.$$

Proof. First one observes the commutation formula

$$(2.7) \quad [\nabla_\alpha, \nabla_\beta] \omega_\zeta^i = (R^M)_{\beta\alpha\zeta}^\delta \omega_\delta^i + (\partial_\beta f^l)(\partial_\alpha f^v)(R^N)_{v\ell k}^i \omega_\zeta^k.$$

Using this we have

$$\begin{aligned} \left(\frac{\partial \tau}{\partial s} \right)^i &= \nabla_s \nabla_\alpha (\partial_\alpha f^i) \\ &= [\nabla_s, \nabla_\alpha] (\partial_\alpha f^i) + \nabla_\alpha (\partial_\alpha \dot{f}^i) \\ &= (\partial_\alpha f^p)(\partial_\alpha f^j)(\dot{f}^q)(R^N)_{pqj}^i - (R^M)_{s\alpha\alpha}^\delta (\partial_\alpha f^i) + \Delta \dot{f}^i \\ &= (\partial_\alpha f^p)(\partial_\alpha f^j)(\dot{f}^q)(R^N)_{pqj}^i + \Delta \dot{f}^i. \end{aligned}$$

The result follows. \square

Proposition 2.6. *Assuming the hypotheses of Proposition 2.4, one has*

$$\begin{aligned} (2.8) \quad \frac{d^2}{ds^2} [\mathcal{F}(f_s, v_s, \theta_s)] &= \frac{\ddot{v}_s}{2} \int_M |Tf_s|^2 \Theta_s dV_g - v_s \int_M \left\langle \ddot{f}_s, \tau_s + \left(\frac{\nabla \Theta_s}{\Theta_s} \lrcorner Tf_s \right) \right\rangle \Theta_s dV_g + \frac{v_s}{2} \int_M |Tf_s|^2 \ddot{\Theta}_s dV_g \\ &\quad - \dot{v}_s \int_M \left(\left\langle \dot{f}_s, \tau_s + \left(\frac{\nabla \Theta_s}{\Theta_s} \lrcorner Tf_s \right) \right\rangle \Theta_s - |Tf_s|^2 \dot{\Theta}_s \right) dV_g \\ &\quad - v_s \int_M \left\langle \dot{f}_s, (\Delta \dot{f}_s + R^N(\dot{f}_s, (Tf_s)_\alpha)(Tf_s)_\alpha) + \nabla_{\frac{\nabla \Theta_s}{\Theta_s}} \dot{f}_s \right\rangle \Theta_s dV_g \\ &\quad - 2v_s \int_M \left\langle \dot{f}_s, \tau_s \left(\frac{\dot{\Theta}_s}{\Theta_s} \right) + \left(\frac{\nabla \dot{\Theta}_s}{\Theta_s} \lrcorner Tf_s \right) \right\rangle \Theta_s dV_g. \end{aligned}$$

Proof. We rewrite (2.5) in the form

$$\begin{aligned} \frac{d}{ds} [\mathcal{F}(f_s, v_s, \theta_s)] &= \left[\frac{\dot{v}_s}{2} \int_M |Tf_s|^2 \Theta_s dV_g \right]_{T_1} - \left[v_s \int_M \left\langle \dot{f}_s, \tau_s + \left(\frac{\nabla \Theta_s}{\Theta_s} \lrcorner Tf_s \right) \right\rangle \Theta_s dV_g \right]_{T_2} \\ &\quad + \left[\frac{v_s}{2} \int_M |Tf_s|^2 \dot{\Theta}_s dV_g \right]_{T_3} \\ &= T_1 - T_2 + T_3. \end{aligned}$$

We differentiate the term T_1 to yield

$$\begin{aligned} \frac{d}{ds} [T_1] &= \frac{\ddot{v}}{2} \int_M |Tf|^2 \Theta dV_g + \dot{v} \int_M \left\langle \nabla \dot{f}, Tf \right\rangle \Theta dV_g + \frac{\dot{v}}{2} \int_M |Tf|^2 \dot{\Theta} dV_g \\ &= \frac{\ddot{v}}{2} \int_M |Tf|^2 \Theta dV_g - \dot{v} \int_M \left\langle \dot{f}, \tau + \left(\frac{\nabla \Theta}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g + \frac{\dot{v}}{2} \int_M |Tf|^2 \dot{\Theta} dV_g. \end{aligned}$$

Using Lemma 2.5, the variation of the second term gives

$$\begin{aligned} \frac{d}{ds} [T_2] &= \dot{v} \int_M \left\langle \dot{f}, \tau + \left(\frac{\nabla \Theta}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g + v \int_M \left\langle \ddot{f}, \tau + \left(\frac{\nabla \Theta}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g \\ &\quad + v \int_M \left\langle \dot{f}, (\Delta \dot{f} + R^N(\dot{f}, Tf_\alpha) Tf_\alpha) + \tau \left(\frac{\dot{\Theta}}{\Theta} \right) + \nabla \dot{f} \left(\frac{\nabla \Theta}{\Theta} \right) + \left(\frac{\nabla \dot{\Theta}}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g. \end{aligned}$$

Finally the variation of the third term is, with an application of integration by parts to the middle quantity,

$$\begin{aligned} \frac{d}{ds} [T_3] &= \frac{\dot{v}}{2} \int_M |Tf|^2 \dot{\Theta} dV_g + v \int_M \left\langle \nabla \dot{f}, Tf \right\rangle \dot{\Theta} dV_g + \frac{v}{2} \int_M |Tf|^2 \ddot{\Theta} dV_g \\ &= \frac{\dot{v}}{2} \int_M |Tf|^2 \dot{\Theta} dV_g - v \int_M \left\langle \dot{f}, \tau \left(\frac{\dot{\Theta}}{\Theta} \right) + \left(\frac{\nabla \dot{\Theta}}{\Theta} \lrcorner Tf \right) \right\rangle \Theta dV_g + \frac{v}{2} \int_M |Tf|^2 \ddot{\Theta} dV_g. \end{aligned}$$

Combining and regrouping one obtains (2.8). \square

2.4. Monotonicity. The following monotonicity formulas from [14] and [16] are essential for the computations of the proof of Theorem 1.1 in §4. We state them, then discuss generalized versions.

Lemma 2.7 ([14] Lemma 3.2). *Let $f_t \in C^\infty(\mathbb{R}^m \times [0, T], N)$ be a solution to (HMHF) with uniformly bounded energy density. Then*

$$\frac{d}{dt} \left[\mathcal{F} \left(f_t, t_0 - t, \frac{|x-x_0|^2}{4(t_0-t)} \right) \right] = -(t_0 - t) \int_{\mathbb{R}^m} \left| \tau_t - \left(\frac{x-x_0}{2(t_0-t)} \lrcorner Tf_t \right) \right|^2 \frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{(4\pi(t_0-t))^{m/2}} dV.$$

Using this and the first variation (cf. Corollary 3.12), Zhang proves the following entropy monotonicity.

Proposition 2.8 ([16] Proposition 4). *Let $f_t \in C^\infty(\mathbb{R}^m \times [0, T], N)$ be a solution to (HMHF) with uniformly bounded energy density. Then the quantity*

$$\lambda(f_t) := \sup_{t \in \mathbb{R}, x \in \mathbb{R}^m} \mathcal{F} \left(f_t, t_0 - t, \frac{|x-x_0|^2}{4(t_0-t)} \right),$$

is non increasing in t .

One may utilize Proposition 2.4 to give a generalization of Lemma 2.7, in the case of general source manifolds M rather than \mathbb{R}^m . This is proven by Hamilton in [9].

Corollary 2.9 ([9] Theorem 1.1, pp.131). *Let $f_t \in C^\infty(\mathbb{R}^m \times [0, T], N)$ be a solution to (HMHF) with uniformly bounded energy, and let v_t and θ_t satisfy (2.3) and (2.4) respectively. Set $\Theta_t := e^{-\theta_t}(4\pi v_t)^{-m/2}$. Then*

$$(2.9) \quad \begin{aligned} \frac{d}{dt} [\mathcal{F}(f_t, v_t, \theta_t)] = & -v_t \int_M \left| \frac{(\nabla \Theta_t) \cdot T f_t}{\Theta_t} + \tau_t \right|^2 \Theta_t dV_g \\ & + v_t \int_M g^{\alpha\beta} g^{\zeta\delta} \langle (T f_t)_\alpha, (T f_t)_\delta \rangle \left(\frac{(\nabla_\zeta \Theta_t)(\nabla_\beta \Theta_t) - \nabla_\zeta \nabla_\beta \Theta_t}{\Theta_t} - \left(\frac{g_{\zeta\beta}}{2v_t} \right) \right) \Theta_t dV_g. \end{aligned}$$

We next state the resultant monotonicity formula, given in Hamilton's paper [9], which is a result of applying the Harnack estimates from [8] to control the last term of (2.9).

Theorem 2.10 ([9] Theorem A, pp.123). *Let $f_t \in C^\infty(M \times [0, T], N)$ solve (HMHF) on $0 \leq t < T$ and v_t and θ_t satisfy (2.3) and (2.4) respectively. Set $\Theta_t := e^{-\theta_t}(4\pi v_t)^{-m/2}$. Then if $\int_M \Theta_t = 1$, the quantity $\mathcal{F}(f_t, v_t, \theta_t)$ is monotone decreasing in t when M is Ricci parallel with weakly positive sectional curvatures. On a general manifold M we have*

$$\mathcal{F}(f_t, v_t, \Theta_t) \leq C_M \left(\mathcal{F}(f_\tau, v_\tau, \Theta_\tau) + (t - \tau) \int_M |T f_0|^2 dV_g \right),$$

for $T - 1 \leq \tau \leq t \leq T$, and $C_M \in \mathbb{R}_{>0}$ a constant depending only on M .

Corollary 2.11. *Let $f_t \in C^\infty(M \times [0, T], N)$ be a solution to (HMHF) with uniformly bounded energy for $t \in [0, T]$. Then we have that*

$$(2.10) \quad \lambda(f_t) \leq C_M \left(\lambda(f_\tau) + (t - \tau) \int_M |T f_0|^2 dV_g \right).$$

for $T - 1 \leq \tau \leq t \leq T$, and $C_M \in \mathbb{R}_{>0}$ a constant depending only on M . Moreover, if M is Ricci parallel with weakly positive sectional curvatures, then $\lambda(f_t)$ is monotone.

Proof. First assume M is Ricci parallel with weakly positive sectional curvatures. Suppose to the contrary that there exists some solution to harmonic map heat flow, f_t such that $\lambda(f_{t_0}) < \lambda(f_{t_1})$ for $t_0 < t_1$. Then we have that there exists some choice of v_{t_1} and θ_{t_1} such that

$$\lambda(f_{t_0}) < \mathcal{F}(f_{t_1}, v_{t_1}, \theta_{t_1}).$$

If we let v_t, θ_t as above satisfying (2.3) and (2.4) with the appropriate final values, then by Corollary 2.9 we have that $\frac{d}{dt} [\mathcal{F}(f_t, v_t, \theta_t)] \leq 0$, so we have that

$$\lambda(f_{t_0}) \geq \mathcal{F}(f_{t_0}, v_{t_0}, \theta_{t_0}) \geq \mathcal{F}(f_{t_1}, v_{t_1}, \theta_{t_1}) > \lambda(f_{t_0}).$$

This is clearly a contradiction; the first result follows.

More generally assume that equality (2.10) is violated for some times $t_0 < t_1$. Then we have that

$$\lambda(f_{t_1}) > C_M \left(\lambda(f_{t_0}) + (t_1 - t_0) \int_M |T f_0|^2 dV_g \right).$$

Then one can choose θ_t, v_t such that

$$\mathcal{F}(f_{t_1}, v_{t_1}, \theta_{t_1}) > C_M \left(\lambda(f_{t_0}) + (t_1 - t_0) \int_M |T f_0|^2 dV_g \right),$$

and moreover using Corollary 2.9 we have that

$$\begin{aligned} \lambda(f_{t_0}) & \geq \frac{1}{C_M} \mathcal{F}(f_{t_1}, v_{t_1}, \theta_{t_1}) - (t_1 - t_0) \int_M |T f_0|^2 dV_g \\ & > \frac{1}{C_M} \left(C_M \lambda(f_{t_0}, v_{t_0}, \theta_{t_0}) + C_M (t_1 - t_0) \int_M |T f|^2 dV_g \right) - (t_1 - t_0) \int_M |T f_0|^2 dV_g \\ & = \lambda(f_{t_0}, v_{t_0}, \theta_{t_0}). \end{aligned}$$

This is a contradiction, so (2.10) follows. \square

3. ENTROPY FOR EUCLIDEAN SOURCE

In the previous section we established a general notion of entropy for maps between Riemannian manifolds, and its relationship to harmonic map flow. As exhibited in [7], type I singularities to harmonic map flow admit blowup solutions which are shrinking solitons, mapping from flat Euclidean space to the given target manifold. The rest of this paper focuses primarily on the study of shrinking solitons, and so we now restrict to the setting where $(M, g) = (\mathbb{R}^m, g_{\text{euc}})$, where g_{euc} denotes the standard flat metric. Also henceforth R will represent the curvature tensor on N .

In this section we review some fundamental properties of solitons and their relationship to the \mathcal{F} -functionals. Many of these results appeared in [16]. We are forced to revisit some of these calculations because the proof of Theorem 1.1 requires us to consider variations of the \mathcal{F} functionals for general basepoints.

3.1. \mathcal{F} -functional revisited. To begin we define recall a simpler version of the \mathcal{F} -functional with designated fixed base point, defined in [16] in analogy with [3]. For $x_0 \in \mathbb{R}^m$ and $t_0 > 0$, define the \mathcal{F}_{x_0, t_0} -functional by

$$\mathcal{F}_{x_0, t_0} : C^\infty(\mathbb{R}^m, N) \rightarrow \mathbb{R},$$

$$\mathcal{F}_{x_0, t_0}(f) = \frac{t_0}{2} \int_M |Tf|^2 \frac{e^{-\frac{|x-x_0|^2}{4t_0}}}{(4\pi t_0)^{m/2}} dV.$$

To connect to the notation of §2, observe that $\mathcal{F}_{x_0, t_0}(f) := \mathcal{F}\left(f, t_0, \frac{|x-x_0|^2}{(4t_0)^{m/2}}\right)$. Furthermore we define the *entropy functional* by

$$\lambda(f) = \sup_{x_0 \in \mathbb{R}^m, t_0 > 0} \mathcal{F}_{x_0, t_0}(f).$$

As established in [16], critical points of the \mathcal{F}_{x_0, t_0} -functional are self-similar shrinking solutions to harmonic map flow.

Definition 3.1. A solution $f_t : \mathbb{R}^m \rightarrow N$ to (HMHF) is called a (x_0, t_0) -self similar solution if it satisfies

$$\tau_t = \left(\frac{x - x_0}{2(t_0 - t)} \right) \lrcorner Tf_t.$$

Due to the self similarity, it is enough to consider the $t = 0$ time slice of a self similar solution, which captures all the information of the flow itself (cf. Proposition 3.3). We call the $t = 0$ slice a (x_0, t_0) -soliton, given by

$$\tau_0 = \left(\frac{x - x_0}{2t_0} \right) \lrcorner Tf_0.$$

The subscript will be omitted when understood. Lastly, taking $(x_0, t_0) = (0, 1)$ we define simply a *soliton*, given by

$$\tau = \left(\frac{x}{2} \right) \lrcorner Tf.$$

We let \mathfrak{S}_{x_0, t_0} denote the space of all (x_0, t_0) -solitons. We define the (x_0, t_0) -soliton operator by

$$\mathbf{S}_{x_0, t_0} : C^\infty(\mathbb{R}^m, N) \rightarrow T^*M \otimes f^*(TN)$$

$$\mathbf{S}_{x_0, t_0}(f) = \tau - \left(\frac{x - x_0}{2t_0} \right) \lrcorner Tf.$$

Note in particular $\mathfrak{S}_{x_0, t_0} = \ker \mathbf{S}_{x_0, t_0}$, and furthermore that \mathbf{S}_{x_0, t_0} is equivalent to \mathbf{S}_Θ for $\Theta = \exp\left(\frac{|x-x_0|^2}{4t_0}\right)$. Furthermore, the set \mathfrak{S}_{x_0, t_0} is equivalent to $\mathfrak{S} := \mathfrak{S}_{0,1}$ via appropriate coordinate shifts, as demonstrated in the following lemma.

Lemma 3.2. *The sets \mathfrak{S}_{x_0, t_0} and \mathfrak{S} are in bijective correspondence for all $x_0 \in \mathbb{R}^m$ and $t_0 > 0$.*

Proof. Suppose that $f \in \mathfrak{S}$. Then set

$$\tilde{f}(x) := f\left(\frac{x-x_0}{t_0^{1/4}}\right).$$

Note that

$$\begin{aligned} \tau_{\tilde{f}}(x) &= \nabla_\alpha \partial_\alpha \left[f^i \left(\frac{x-x_0}{t_0^{1/4}} \right) \right] \\ &= \frac{1}{\sqrt{t_0}} \tau_f \left(\frac{x-x_0}{\sqrt{t_0}} \right) \\ &= \frac{1}{\sqrt{t_0}} T f_\alpha \left(\frac{(x-x_0)^\alpha}{2\sqrt{t_0}} \right) \\ &= T f_\alpha \left(\frac{x-x_0}{2t_0} \right)^\alpha. \end{aligned}$$

Thus we have defined a mapping to \mathfrak{S}_{x_0, t_0} . The inverse map sends $f \in \mathfrak{S}_{x_0, t_0}$ to $\bar{f}(x) := f(x_0 + \sqrt{t_0}x)$, as is easily checked. \square

There are a multitude of ways to characterize solitons; we state and explicitly demonstrate some in the following proposition. We prove the proposition in a series of lemmas, marked following each corresponding item. Item (3) is postponed to later in the section.

Proposition 3.3. *A smooth map $f : \mathbb{R}^m \rightarrow N$ is a soliton if and only if any of the following holds.*

- (1) (Lemma 3.4). *The one-parameter family $f_t := \sqrt{-t}f \in C^\infty(\mathbb{R}^m, N)$ satisfies (HMHF).*
- (2) (Lemma 3.5) *f is harmonic with respect to the conformal metric $g := e^{-\frac{|x|^2}{4(m-2)}} g_{\text{euc}}$.*
- (3) (Corollary 3.13) *f is a critical point for the $\mathcal{F}_{0,1}$ -functional.*

Lemma 3.4. *A smooth map $f : \mathbb{R}^m \rightarrow N$ is a soliton if and only if the one-parameter family*

$$(3.1) \quad f_t(x) = f_{-1} \left(\frac{x}{\sqrt{-t}} \right), \quad t < 0,$$

is a solution to (HMHF).

Proof. First assume that (3.1) holds. Take, for $\kappa \in \mathbb{R}_{>0}$,

$$f(x, t) := f(\kappa x, \kappa^2 t).$$

Differentiating with respect to κ and evaluating at $\kappa = 1$ yields

$$\begin{aligned} 0 &= \frac{\partial}{\partial \kappa} (f_\kappa - f)|_{\kappa=1} \\ &= x^\alpha \partial_\alpha f_\kappa + 2t\kappa \partial_t f_\kappa|_{\kappa=1} \\ &= x \lrcorner T f + 2t \partial_t f. \end{aligned}$$

Therefore since $\partial_t f = \tau$ we have the desired soliton equation. Consequently if we take $\kappa := (-t)^{-1/2}$ we have the desired result.

Next assume \bar{f}_t is another solution to the self similar solution equation with $\bar{f}_{-1} \equiv f$. Take $E : N \rightarrow \mathbb{R}^n$ to be some smooth embedding. Define $\psi_t := E(f_t) - E(\bar{f}_t)$. Then by the chain rule, for each t we have that ψ_t is in the kernel of the operator

$$\Psi : C^\infty(\mathbb{R}^m \times \mathbb{R}_{\leq 0}, \mathbb{R}^n) \rightarrow T\mathbb{R}^n : \psi \mapsto \frac{\partial \psi}{\partial t} + T\psi_t \left(\frac{x}{2t} \right).$$

We verify that for any $s \in \mathbb{R}_{\leq 0}$ the hypersurface $\mathbb{R}^m \times \{s\}$ is non-characteristic with respect to the operator Ψ , which is equivalent to demonstrating that Ψ is non degenerate in the transverse direction of the boundary of $\mathbb{R}^m \times \{s\}$, namely,

$$\langle \sigma[\Psi], \partial_t \rangle \neq 0,$$

where here $\sigma[\Psi]$ denotes the symbol of Ψ . Now since

$$(\sigma[\Psi](\psi)) = \xi_t \psi + \xi_x \psi \left(\frac{x}{2t} \right),$$

then it follows that

$$\langle \sigma[\Psi], \xi_t \rangle = |\xi_t|^2 \neq 0.$$

Thus, by Holmgren's Uniqueness Theorem (cf. [15] pp.433) there exists some $\epsilon > 0$ such that on $\mathbb{R}^m \times [s - \epsilon, s + \epsilon]$, we have $\Psi(\psi) \equiv 0$. Therefore the set

$$\mathcal{T} := \{\theta \in \mathbb{R}_{\leq 0} : \psi_\theta = 0\}$$

is open. Since this set is closed (it is $\Psi^{-1}(\{0\})$ and Ψ is continuous) due to the connectedness of $\mathbb{R}_{\leq 0}$ we conclude that $\mathcal{T} = (-\infty, 0)$, therefore $f_t \equiv \bar{f}_t$, as desired. The result follows. \square

Lemma 3.5. *A map $f : \mathbb{R}^m \rightarrow N$ is a soliton if and only if f is a harmonic map with respect to the metric $e^{\frac{-|x|^2}{4(m-2)}} g_{\text{euc}}$.*

Proof. We begin with a general computation of how the tension field changes under a conformal change of the base metric. Take $g_{\alpha\beta} := e^{2\psi} \delta_{\alpha\beta}$ and then recall the formula for the conformally changed Christoffel symbols

$$(\Gamma^g)_{\alpha\beta}^\zeta = -\delta^{\zeta\omega} (\partial_\omega \psi) \delta_{\alpha\beta} + (\partial_\alpha \psi) \delta_\beta^\zeta + (\partial_\beta \psi) \delta_\alpha^\zeta.$$

Since one has

$$\tau_g^i = f_{\alpha\alpha}^i - (\Gamma^g)_{\alpha\alpha}^\zeta f_\zeta^i + f_\alpha^k f_\alpha^j \Gamma_{kj}^i,$$

it follows that

$$\begin{aligned} \tau_g^i &= \tau^i - \left(-\delta^{\zeta\omega} (\partial_\omega \psi) \delta_{\alpha\alpha} + (\partial_\alpha \psi) \delta_\alpha^\zeta + (\partial_\alpha \psi) \delta_\alpha^\zeta \right) f_\zeta^i \\ &= \tau^i + (\partial_\omega f^i) (\partial_\omega \psi) m - (\partial_\alpha \psi) (\partial_\alpha f^i) - (\partial_\alpha \psi) (\partial_\alpha f^i) \\ &= \tau^i + (m-2)((\nabla \psi) \lrcorner T f)^i. \end{aligned}$$

Choosing $\psi = \frac{-|x|^2}{4(m-2)}$ the second term becomes

$$\begin{aligned} (m-2)(\nabla \psi \lrcorner T f) &= -(m-2) \left(\frac{x}{2(m-2)} \lrcorner T f \right) \\ &= -\left(\frac{x}{2} \lrcorner T f \right). \end{aligned}$$

Applying this we see that

$$\tau_g^i = \tau^i - \left(\frac{x}{2} \lrcorner T f \right)^i.$$

The result follows. \square

3.2. Preliminary identities. In this subsection we establish some basic identities needed for the proof of Theorem 1.1. If we define the quantity

$$G_{x_0, t_0}(x, t) := \frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{(4\pi(t_0-t))^{\frac{m}{2}}}.$$

then G_{x_0, t_0} is the heat kernel of \mathbb{R}^m and thus satisfies the backwards heat flow (BHF). When the basepoint is understood we will use the notation $G_0(x) = G_{x_0, t_0}(x, 0)$.

Definition 3.6. Suppose $f : \mathbb{R}^m \rightarrow N$. Then f has *polynomial energy density growth* if there exists a polynomial p such that for all $x \in \mathbb{R}^m$, one has $|Tf(x)| \leq p(|x|)$. Also, we say that f has *polynomial energy growth* if there exists a polynomial p such that

$$\int_{B_{|x|}} |Tf|^2 dV \leq p(|x|).$$

Note the assumption of polynomial energy growth needs to be made in various lemmas below, to justify the applications of dominated convergence. We first prove a generalization of Lemma 3 of [16] with varying base points for a (χ, σ) -soliton f and the heat kernel G_0 . We require this generalization for the proof of Theorem 1.1.

Lemma 3.7. *Let $f \in \mathfrak{S}_{\chi, \sigma}$ satisfy polynomial energy growth, and let $\varphi = \varphi^\alpha \partial_\alpha$ be a vector field on M such that $|\varphi|^2 G_0 \in L^\infty(\mathbb{R}^m)$. Then*

$$(3.2) \quad \begin{aligned} \int_{\mathbb{R}^m} \varphi^\alpha (x - x_0)^\alpha |Tf|^2 G_0 dV &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle (\partial_\beta \varphi^\alpha) G_0 dV + 2t_0 \int_{\mathbb{R}^m} (\partial_\alpha \varphi^\alpha) |Tf|^2 G_0 dV \\ &\quad + 2 \int_{\mathbb{R}^m} \left(\frac{t_0}{\sigma} \chi - x_0 + x \left(1 - \frac{t_0}{\sigma} \right) \right)^\beta \langle Tf_\beta, Tf_\alpha \rangle \varphi^\alpha G_0 dV. \end{aligned}$$

Proof. First observe that

$$(3.3) \quad \frac{\partial G_0}{\partial x^\alpha} = -\frac{(x - x_0)^\alpha}{2t_0} G_0.$$

Let $\eta \in C_c^\infty(\mathbb{R}^m)$ and $\varphi = \varphi^\alpha \partial_\alpha$ be some smooth vector field on \mathbb{R}^m . Then integrating by parts yields

$$(3.4) \quad \begin{aligned} \int_{\mathbb{R}^m} \varphi^\alpha (x - x_0)^\alpha |Tf|^2 \eta G_0 dV &= -2t_0 \int_{\mathbb{R}^m} \varphi^\alpha |Tf|^2 (\partial_\alpha G_0) \eta dV \\ &= 2t_0 \int_{\mathbb{R}^m} (\partial_\alpha (\eta \varphi^\alpha) |Tf|^2 + \varphi^\alpha (\partial_\alpha |Tf|^2) \eta) G_0 dV. \end{aligned}$$

We consider the following manipulation, involving integration by parts once more, in order to obtain an identity for the second term above.

$$\begin{aligned} 4t_0 \int_{\mathbb{R}^m} \langle \tau, Tf_\alpha \rangle \varphi^\alpha \eta G_0 dV &= 4t_0 \int_{\mathbb{R}^m} \langle \nabla_\beta Tf_\beta, Tf_\alpha \rangle \varphi^\alpha \eta G_0 dV \\ &= 4t_0 \int_{\mathbb{R}^m} (\partial_\beta \langle Tf_\beta, Tf_\alpha \rangle - \langle Tf_\beta, \nabla_\beta Tf_\alpha \rangle) \varphi^\alpha \eta G_0 dV \\ &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle \partial_\beta [\varphi^\alpha \eta G_0] dV - 4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, \nabla_\alpha Tf_\beta \rangle \varphi^\alpha \eta G_0 dV \\ &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle \left(\partial_\beta [\varphi^\alpha \eta] G_0 - \frac{(x - x_0)^\beta}{2t_0} \eta \varphi^\alpha G_0 \right) dV \\ &\quad - 2t_0 \int_{\mathbb{R}^m} \partial_\alpha |Tf|^2 \varphi^\alpha \eta G_0 dV. \end{aligned}$$

We therefore conclude that

$$(3.5) \quad \begin{aligned} 2t_0 \int_{\mathbb{R}^m} \partial_\alpha |Tf|^2 \eta G_0 dV &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle \left(\partial_\beta [\varphi^\alpha \eta] G_0 - \frac{(x - x_0)^\beta}{2t_0} \eta \varphi^\alpha G_0 \right) dV \\ &\quad - 4t_0 \int_{\mathbb{R}^m} \langle \tau, Tf_\alpha \rangle \varphi^\alpha \eta G_0 dV. \end{aligned}$$

Applying this identity to (3.4) we conclude that

(3.6)

$$\begin{aligned} \int_{\mathbb{R}^m} \varphi^\alpha(x-x_0)^\alpha |Tf|^2 \eta G_0 dV &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle \left(\partial_\beta [\varphi^\alpha \eta] G_0 - \frac{(x-x_0)^\beta}{2t_0} \varphi^\alpha \eta G_0 \right) dV \\ &\quad + 2t_0 \int_{\mathbb{R}^m} \partial_\alpha [\varphi^\alpha \eta] |Tf|^2 G_0 dV - 4t_0 \int_{\mathbb{R}^m} \langle \tau, Tf_\alpha \rangle \varphi^\alpha \eta G_0 dV. \end{aligned}$$

We let η_R be a cut off function with support within B_R which cuts off to zero linearly between B_R and B_{R+1} . Setting $\eta = \eta_R$ above in (3.6) and sending $R \rightarrow \infty$, it follows from the Dominated Convergence Theorem that we can remove η from the quantities above. Since $f \in \mathfrak{S}_{\chi, \sigma}$ we replace $\tau = \frac{(x-\chi)}{2\sigma}$ accordingly

$$\begin{aligned} \int_{\mathbb{R}^m} \varphi^\alpha(x-x_0)^\alpha |Tf|^2 G_0 dV &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle (\partial_\beta \varphi^\alpha) G_0 dV + 2t_0 \int_{\mathbb{R}^m} (\partial_\alpha \varphi^\alpha) |Tf|^2 G_0 dV \\ &\quad + 2 \int_{\mathbb{R}^m} \left(\frac{t_0}{\sigma} \chi - x_0 + x \left(1 - \frac{t_0}{\sigma} \right) \right)^\beta \langle Tf_\beta, Tf_\alpha \rangle \varphi^\alpha G_0 dV. \end{aligned}$$

The result follows. \square

Corollary 3.8. *Suppose that $f \in \mathfrak{S}_{\chi, \sigma}$. Then the following identities hold.*

(a)

$$\begin{aligned} \int_{\mathbb{R}^m} |x-x_0|^2 |Tf|^2 G_0 dV &= 2 \int_{\mathbb{R}^m} \left(|(x-x_0) \lrcorner Tf|^2 - \frac{t_0}{\sigma} \langle (x-\chi) \lrcorner Tf, (x-x_0) \lrcorner Tf \rangle \right) G_0 dV \\ &\quad + 2t_0 \int_{\mathbb{R}^m} (m-2) |Tf|^2 G_0 dV. \end{aligned}$$

(b)

$$\int_{\mathbb{R}^m} (x-x_0)^\gamma |Tf|^2 G_0 dV = 2 \int_{\mathbb{R}^m} \left\langle \left(\frac{t_0}{\sigma} \chi - x_0 + x \left(1 - \frac{t_0}{\sigma} \right) \lrcorner Tf \right), Tf_\gamma \right\rangle G_0 dV.$$

Proof. Identity (a) follows by setting $\varphi^\alpha := (x-x_0)^\alpha$, while identity (b) follows by setting $\varphi^\alpha := \delta^{\alpha\gamma}$. \square

The following corollary is a result of coinciding base points of the heat kernel G_0 and f , and follows from choice of appropriate vector fields.

Corollary 3.9 ([16], Lemma 3). *Suppose that $f \in \mathfrak{S}$, $\gamma \in [1, m] \cap \mathbb{N}$, and $\zeta = \zeta^\alpha \partial_\alpha$ is some constant vector field on \mathbb{R}^m . Then the following identities hold.*

- (a) $\int_{\mathbb{R}^m} \left(\frac{2-m}{4} + \frac{|x-x_0|^2}{8t_0} \right) |Tf|^2 G_0 dV = 0,$
- (b) $\int_{\mathbb{R}^m} (x-x_0)^\gamma |Tf|^2 G_0 dV = 0,$
- (c) $\int_{\mathbb{R}^m} |x-x_0|^4 |Tf|^2 G_0 dV = \int_{\mathbb{R}^m} (4m(m-2)t_0^2 |Tf|^2 - 32t_0^3 |\tau|^2) G_0 dV,$
- (d) $\int_{\mathbb{R}^m} |x-x_0|^2 \langle \zeta, x-x_0 \rangle |Tf|^2 G_0 dV = \int_{\mathbb{R}^m} \langle \zeta \lrcorner Tf, \tau \rangle G_0 dV = 0,$
- (e) $\int_{\mathbb{R}^m} \langle \zeta, x-x_0 \rangle^2 |Tf|^2 G_0 dV = 2t_0 \int_{\mathbb{R}^m} (|\zeta|^2 |Tf|^2 - 2|\zeta \lrcorner Tf|^2) G_0 dV.$

Proof. The lemma follows by choosing various test functions, indicated below.

- (a) This follows by setting $\varphi^\alpha := \frac{(x-x_0)^\alpha}{8t_0}$.
- (b) This follows by setting $\varphi^\alpha := \delta^{\alpha\gamma}$.
- (c) We set $\varphi^\alpha := |x-x_0|^2 (x-x_0)^\alpha$. Then we have that

$$\partial_\beta \varphi^\alpha = 2(x-x_0)^\beta (x-x_0)^\alpha + \delta_\beta^\alpha |x-x_0|^2.$$

and obtain

$$\begin{aligned} \int_{\mathbb{R}^m} |x - x_0|^4 |Tf|^2 G_0 dV &= -8t_0 \int_{\mathbb{R}^m} |Tf(x - x_0)|^2 G_0 dV - 4t_0 \int_{\mathbb{R}^m} |Tf|^2 |x - x_0|^2 G_0 dV \\ &\quad + 2t_0 \int_{\mathbb{R}^m} ((2 + m)|x - x_0|^2) |Tf|^2 G_0 dV \\ &= -32t_0^3 \int_{\mathbb{R}^m} |\tau|^2 G_0 dV + 2mt_0 \int_{\mathbb{R}^m} (|x - x_0|^2) |Tf|^2 G_0 dV. \end{aligned}$$

applying part (a) to the second term yields

$$\int_{\mathbb{R}^m} |x - x_0|^4 |Tf|^2 G_0 dV = -32t_0^3 \int_{\mathbb{R}^m} |\tau|^2 G_0 dV + 4m(m - 2)t_0^2 \int_{\mathbb{R}^m} |Tf|^2 G_0 dV.$$

(d) First setting $\varphi^\alpha := |x - x_0|^2 \zeta^\alpha$ and applying (b) we have

$$(3.7) \quad \int_{\mathbb{R}^m} |x - x_0|^2 \langle (x - x_0), \zeta \rangle |Tf|^2 G_0 dV = -8t_0 \int_{\mathbb{R}^m} \langle (x - x_0) \lrcorner Tf, \zeta \lrcorner Tf \rangle G_0 dV.$$

Next we will consider $\varphi^\alpha := \langle \zeta, x - x_0 \rangle (x - x_0)^\alpha$. Note that

$$\partial_\beta [\zeta^\mu (x - x_0)^\mu (x - x_0)^\alpha] = \zeta^\mu \delta_\beta^\mu (x - x_0)^\alpha + \zeta^\mu (x - x_0)^\mu \delta_\beta^\alpha = \zeta^\beta (x - x_0)^\alpha + \langle \zeta, x - x_0 \rangle \delta_\beta^\alpha.$$

Applying this identity, and using (b) once more we obtain

$$\begin{aligned} (3.8) \quad \int_{\mathbb{R}^m} |x - x_0|^2 \langle (x - x_0), \zeta \rangle |Tf|^2 G_0 dV &= -4t_0 \int_{\mathbb{R}^m} \langle Tf_\beta, Tf_\alpha \rangle (\zeta^\beta (x - x_0)^\alpha + \langle \zeta, x - x_0 \rangle \delta_\beta^\alpha) G_0 dV \\ &\quad + 2t_0 \int_{\mathbb{R}^m} ((1 + m) \langle \zeta, (x - x_0) \rangle) |Tf|^2 G_0 dV \\ &= -4t_0 \int_{\mathbb{R}^m} \langle \zeta \lrcorner Tf, (x - x_0) \lrcorner Tf \rangle G_0 dV. \end{aligned}$$

Comparing equalities (3.7) and (3.8) we conclude the desired result.

(e) This follows by setting $\varphi^\alpha := \langle \zeta, x - x_0 \rangle \zeta^\alpha$.

The results follow. \square

We next observe that as a consequence of the above identities, the harmonic map heat flow with source of dimension $m = 2$ cannot exhibit type I singularities.

Proposition 3.10. *Let $f_t : M^m \rightarrow N$ be a smooth solution to harmonic map heat flow which exists on a maximal time interval $[0, t_0)$ with $t_0 < \infty$. If $m = 2$ then*

$$\lim_{t \rightarrow t_0} (t_0 - t) |Tf_t| = \infty.$$

Moreover, any soliton on \mathbb{R}^m for $m \leq 2$ is constant.

Proof. Suppose to the contrary there exists some $C \in \mathbb{R}$ such that $\lim_{t \rightarrow t_0} (t_0 - t) |Tf_t| \leq C$. By [7] one may construct a type I blowup limit f_∞ which is a self similar solution and thus its time $t = 0$ slice with $t_0 = 1$ is a non constant soliton. By Corollary 3.9 (a), since $m = 2$ one has that

$$\int_{\mathbb{R}^m} \left(\frac{|x|^2}{8} \right) |Tf|^2 G_0 dV = 0.$$

Therefore f_∞ is constant, but this is a contradiction. Thus, the result follows. \square

3.3. Variation identities. For the following computations we will consider the quantities f, x, t to be varying by a parameter s . Often notational dependency on s will be suppressed except for on the base point variables (x_s, t_s) . We will first lay out some fundamental variation identities which will be applied to the formulas of Propositions 2.4 and 2.6 to demonstrate the identities specific to the Euclidean source case. We then evaluate at a soliton in order to derive the L^f operator, a quantity whose negative spectrum characterizes the stability of its corresponding soliton f .

Lemma 3.11. *The following identities hold.*

$$\begin{aligned}\frac{dG_s}{ds} &= \mathbf{g}_s G_s, \\ \frac{d^2 G_s}{ds^2} &= \left(\frac{d\mathbf{g}_s}{ds} \right) G_s + (\mathbf{g}_s)^2 G_s,\end{aligned}$$

where

$$(3.9) \quad \mathbf{g}_s := \left(-\frac{m\dot{t}_s}{2t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right).$$

Proof. Using chain rule, and observing that $G_s = \frac{e^{-\frac{|x-x_s|^2}{4t_s}}}{(4\pi t_s)^{m/2}}$ and applying 3.3 we have

$$\begin{aligned}\frac{dG_s}{ds} &= \frac{\partial G_s}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial s} + \frac{\partial G_s}{\partial t} \frac{\partial t}{\partial s} \\ &= -\frac{\langle x-x_s, \dot{x}_s \rangle}{2t_s} \frac{e^{-\frac{|x-x_s|^2}{4t_s}}}{(4\pi t_s)^{m/2}} + \left(-\frac{\dot{t}_s}{4t_s^2} \right) \frac{e^{-\frac{|x-x_s|^2}{4t_s}}}{(4\pi t_s)^{m/2}} - \frac{m\dot{t}_s}{2} \frac{e^{-\frac{|x-x_s|^2}{4t_s}}}{(4\pi)^{m/2} t_s^{m/2+1}} \\ &= \left(-\frac{m\dot{t}_s}{2t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s \\ &= \mathbf{g}_s G_s.\end{aligned}$$

This gives the first identity. Differentiating again we have that

$$\begin{aligned}\frac{d^2 G_s}{ds^2} &= \frac{d}{ds} [\mathbf{g}_s G_s] \\ &= \left(\frac{d\mathbf{g}_s}{ds} + \mathbf{g}_s^2 \right) G_s.\end{aligned}$$

The result follows. \square

Next we compute some identities operations performed on \mathbf{g}_s (cf. (3.9)) which will be used for the following variations.

$$(3.10) \quad \frac{\partial \mathbf{g}_s}{\partial x^\alpha} = \left(\frac{\dot{t}_s}{2t_s} (x-x_s)^\alpha + \frac{\dot{x}_s^\alpha}{2t_s} \right)$$

$$(3.11) \quad \frac{d\mathbf{g}_s}{ds} = -\frac{m}{2} \left(\frac{\ddot{t}_s}{t_s} - \frac{\dot{t}_s^2}{t_s^2} \right) + \frac{|x-x_s|^2}{4} \left(\frac{\ddot{t}_s}{t_s^2} - \frac{2\dot{t}_s^2}{t_s^3} \right) - \frac{\dot{t}_s \langle \dot{x}_s, x-x_s \rangle}{t_s^2} + \frac{\langle \ddot{x}_s, x-x_s \rangle}{2t_s} - \frac{|\dot{x}_s|^2}{2t_s}$$

$$\begin{aligned}(3.12) \quad \mathbf{g}_s^2 &= \left(-\frac{m\dot{t}_s}{2t_s} + \frac{\dot{t}_s|x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right)^2 \\ &= \frac{m^2 \dot{t}_s^2}{4t_s^2} - \frac{m\dot{t}_s^2|x-x_s|^2}{4t_s^3} - \frac{m\dot{t}_s \langle \dot{x}_s, x-x_s \rangle}{2t_s^2} + \frac{\dot{t}_s|x-x_s|^2 \langle \dot{x}_s, x-x_s \rangle}{4t_s^3} \\ &\quad + \frac{\dot{t}_s^2|x-x_s|^4}{16t_s^4} + \frac{\langle \dot{x}_s, x-x_s \rangle^2}{4t_s^2}.\end{aligned}$$

Corollary 3.12 ([16] Proposition 3). *Let f_s , x_s , t_s denote the variations of f , x_0 , and t_0 respectively, and set*

$$\dot{t}_s = \frac{dt_s}{ds}, \quad \dot{x}_s = \frac{dx_s}{ds}, \quad \dot{f}_s = \frac{df_s}{ds}.$$

Assuming that $f_s \in H_{loc}^1(\mathbb{R}^m, N)$ and \dot{f}_s satisfies the integrability condition

$$\int_{\mathbb{R}^m} \left(|\dot{f}_s|^2 + |\nabla \dot{f}_s|^2 + |x - x_s|^2 |Tf_s|^2 + |\tau_s|^2 \right) G_s dV < +\infty.$$

Then

$$(3.13) \quad \begin{aligned} \frac{d}{ds} [\mathcal{F}_{x_s, t_s}(f_s)] &= -t_s \int_{\mathbb{R}^m} \left\langle \dot{f}_s, S_{x_s, t_s}(f_s) \right\rangle G_s dV \\ &\quad + \dot{t}_s \int_{\mathbb{R}^m} \left(\frac{2-m}{4} + \frac{|x - x_s|^2}{8t_s} \right) |Tf_s|^2 G_s dV \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^m} \langle \dot{x}_s, x - x_s \rangle |Tf_s|^2 G_s dV. \end{aligned}$$

Proof. By appealing to Proposition 2.4 and apply the identities of Lemma 3.11 we obtain

$$\frac{d}{ds} [\mathcal{F}_{x_s, t_s}(f_s)] = \frac{1}{2} \int_{\mathbb{R}^m} (\dot{t}_s + t_s \mathbf{g}_s) |Tf_s|^2 G_s dV - t_s \int_{\mathbb{R}^m} \left\langle \dot{f}_s, S_{x_s, t_s}(f_s) \right\rangle G_s dV.$$

Simply applying the definition of \mathbf{g}_s , (3.9), and rearranging terms gives the result. \square

One can conclude the following statement from this formulation combined with the identities (a) and (b) of Corollary 3.9 applied to (3.13).

Corollary 3.13 ([16] Corollaries 1, 2). *The point (∇, x_0, t_0) is a critical point of the \mathcal{F} -functional if and only if ∇ is an (x_0, t_0) -soliton.*

Corollary 3.14. *Assuming that $f_s \in H_{loc}^1(\mathbb{R}^m \times I, N)$ and for all $s \in I$, \dot{f}_s satisfies the integrability condition*

$$\int_{\mathbb{R}^m} \left(|\dot{f}_s|^2 + |\nabla \dot{f}_s|^2 + |x - x_s|^2 |Tf_s|^2 + |\tau_s|^2 \right) G_s dV < +\infty.$$

Then

$$(3.14) \quad \begin{aligned} \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(f_s)] &= -t_s \int_{\mathbb{R}^m} \left\langle \dot{f}_s, \Delta \dot{f}_s + R^N(\dot{f}_s, (Tf_s)_\alpha)(Tf_s)_\alpha - \nabla_{\frac{x-x_s}{2t_s}} \dot{f}_s \right\rangle G_s dV \\ &\quad - \int_{\mathbb{R}^m} \left\langle \dot{f}_s, (2t_s \mathbf{g} + \dot{t}_s) S_{x_s, t_s}(f_s) + (\dot{t}_s(x - x_s) + \dot{x}) \lrcorner Tf_s \right\rangle G_s dV \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^m} (t_s (\dot{\mathbf{g}}_s + \mathbf{g}_s^2) + 2\dot{t}_s \mathbf{g} + \ddot{t}_s) |Tf_s|^2 G_s dV_g - t_s \int_{\mathbb{R}^m} \left\langle \ddot{f}_s, S_{x_s, t_s}(f_s) \right\rangle G_s dV. \end{aligned}$$

Proof. We start by applying the identities of Lemma 3.11 and (3.9) directly to the formula of Proposition 2.6.

$$(3.14) \quad \begin{aligned} \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(f_s)] &= \frac{\ddot{t}_s}{2} \int_{\mathbb{R}^m} |Tf_s|^2 G_s dV - t_s \int_{\mathbb{R}^m} \left\langle \ddot{f}_s, S_{x_s, t_s}(f_s) \right\rangle G_s dV + \frac{t_s}{2} \int_{\mathbb{R}^m} |Tf_s|^2 (\dot{\mathbf{g}}_s + \mathbf{g}_s^2) G_s dV \\ &\quad - \dot{t}_s \int_{\mathbb{R}^m} \left(\left\langle \dot{f}_s, S_{x_s, t_s}(f) \right\rangle - |Tf_s|^2 \mathbf{g}_s \right) G_s dV \\ &\quad - t_s \int_{\mathbb{R}^m} \left\langle \dot{f}_s, (\Delta \dot{f}_s + R^N(\dot{f}_s, (Tf_s)_\alpha)(Tf_s)_\alpha - \nabla_{\frac{x-x_s}{2t_s}} \dot{f}_s) \right\rangle G_s dV \\ &\quad - 2t_s \int_{\mathbb{R}^m} \left\langle \dot{f}_s, \tau_s \mathbf{g}_s + \left(\frac{\nabla(\mathbf{g}_s G_s)}{G_s} \lrcorner Tf_s \right) \right\rangle G_s dV. \end{aligned}$$

We expand the quantity of the last line.

$$\begin{aligned}
\int_{\mathbb{R}^m} \left\langle \dot{f}, \tau \mathbf{g} + \left(\frac{\nabla(\mathbf{g}G)}{G} \right) \lrcorner Tf \right\rangle G_s dV &= \int_{\mathbb{R}^m} \left\langle \dot{f}, \tau \mathbf{g} + Tf \left((\nabla \mathbf{g}) + \mathbf{g} \left(\frac{\nabla G}{G} \right) \right) \right\rangle G_s dV \\
&= \int_{\mathbb{R}^m} \left\langle \dot{f}, \tau \mathbf{g} + \left(\frac{\dot{t}_s}{2t_s} (x - x_s) + \frac{\dot{x}_s}{2t_s} - \mathbf{g} \left(\frac{x - x_s}{2t_s} \right) \right) \lrcorner Tf \right\rangle G_s dV \\
&= \int_{\mathbb{R}^m} \left\langle \dot{f}, \mathbf{g} S_{x_s, t_s}(f) + \frac{1}{2t_s} (\dot{t}_s (x - x_s) + \dot{x}) \lrcorner Tf \right\rangle G_s dV.
\end{aligned}$$

Then we regroup terms and obtain the result. \square

Evaluating the previous variation at a soliton allows us to conclude the formula for the L^f operator, which will be key in the following computations. This will be given by

$$\begin{aligned}
(3.15) \quad L^f : \Gamma(f^*(TN)) &\rightarrow \Gamma(f^*(TN)) \\
: X &\mapsto \left[-\Delta X - R(X, Tf_\alpha) Tf_\alpha + \frac{\nabla_{(x-x_0)} X}{2t_0} \right].
\end{aligned}$$

We also set

$$(3.16) \quad W_f^{2,2} := \left\{ X \in \Gamma(f^*TN) : \int_{\mathbb{R}^m} \left(|X|^2 + |\nabla X|^2 + |L^f X|^2 \right) G_0 dV_g \right\}.$$

Proposition 3.15 ([16] Proposition 5). *Let f be a soliton and $\dot{f} = \dot{f}^\alpha \partial_\alpha$ be some vector field on N such that $\dot{f} \in W_f^{2,2}$. Let f_s be some one parameter family with $f_0 = f$, and \dot{t}_0 and \dot{x}_0 basepoint variations. Then*

$$\begin{aligned}
\mathcal{F}''(\dot{t}, \dot{x}, \dot{f}) &= \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(f_s)]|_{s=0} \\
&= t_0 \int_{\mathbb{R}^m} \left\langle \dot{f}, L^f \dot{f} - 2\dot{t}_0 \tau - (\dot{x}_0 \lrcorner Tf) \right\rangle G_0 dV - \int_{\mathbb{R}^m} \left(\dot{t}_0^2 |\tau|^2 + \frac{1}{2} |\dot{x}_0 \lrcorner Tf|^2 \right) G_0 dV.
\end{aligned}$$

Proof. Since $S_{x_0, t_0}(f) = 0$ and thus $Tf_\alpha(x - x_0)^\alpha = 2t_0 \tau$,

$$\begin{aligned}
(3.17) \quad \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(f)] \Big|_{t=0} &= -t_s \int_{\mathbb{R}^m} \left\langle \dot{f}, \Delta \dot{f} + R(\dot{f}, Tf_\alpha) Tf_\alpha - \frac{1}{2t_0} \nabla_{(x-x_0)} \dot{f} \right\rangle G dV \\
&\quad - \int_{\mathbb{R}^m} \left\langle \dot{f}, 2\dot{t}_0 \tau + \dot{x}_0 \lrcorner Tf \right\rangle G dV \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^m} (t_0(\dot{\mathbf{g}} + \mathbf{g}^2) + \ddot{t}_0 + 2\dot{t}_0 \mathbf{g}) |Tf|^2 G dV.
\end{aligned}$$

We expand, marking with the applicable identities of Corollary 3.8,

$$\begin{aligned}
\dot{\mathbf{g}} + \mathbf{g}^2 &= \left(-\frac{m\ddot{t}_0}{2t_0} + \frac{m\dot{t}_0^2}{2t_0^2} + \frac{m^2\dot{t}_0^2}{4t_0^2} \right) + \left(\frac{|x - x_0|^2}{4t_0} \left(\frac{\ddot{t}_0}{t_0} - \frac{m\dot{t}_0^2}{t_0^2} - \frac{2\dot{t}_0^2}{t_0^2} \right) \right)_a + \left(\langle \dot{x}_0, x - x_0 \rangle \left(-\frac{\dot{t}_0}{t_0^2} - \frac{m\dot{t}_0}{2t_0^2} \right) \right)_b \\
&\quad + \left(|x - x_0|^2 \langle \dot{x}_0, x - x_0 \rangle \left(\frac{\dot{t}_0}{4t_0^3} \right) \right)_d + \left(\langle \ddot{x}_0, x - x_0 \rangle \frac{1}{2t_0} \right)_b + \left(|\dot{x}_0|^2 \left(\frac{-1}{2t_0} \right) \right) \\
&\quad + \left(|x - x_0|^4 \left(\frac{\dot{t}_0^2}{16t_0^4} \right) \right)_c + \left(\langle \dot{x}_0, x - x_0 \rangle^2 \left(\frac{1}{4t_0^2} \right) \right)_e.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^m} (\dot{\mathbf{g}} + \mathbf{g}^2) |Tf|^2 G dV &= \int_{\mathbb{R}^m} \left(-\frac{m\ddot{t}_0}{2t_0} + \frac{m\dot{t}_0^2}{2t_0^2} + \frac{m^2\dot{t}_0^2}{4t_0^2} \right) |Tf|^2 G dV \\
&\quad + \int_{\mathbb{R}^m} \left(\frac{(m-2)}{2} \left(\frac{\dot{t}_0}{t_0} - \frac{m\dot{t}_0^2}{t_0^2} - \frac{2\dot{t}_0^2}{t_0^2} \right) \right) |Tf|^2 G dV \\
&\quad - \int_{\mathbb{R}^m} \left(\frac{|\dot{x}_0|^2}{2t_0} \right) |Tf|^2 G dV + \left(\frac{1}{4t_0^2} \right) 2t_0 \int_{\mathbb{R}^m} \left(|\dot{x}_0|^2 |Tf|^2 - 2|\dot{x}_0 \lrcorner Tf|^2 \right) G dV \\
&\quad + \left(\frac{\dot{t}_0^2}{16t_0^2} \right) \left(\int_{\mathbb{R}^m} (4m(m-2)|Tf|^2 - 32t_0|\tau|^2) G dV \right) \\
&= \int_{\mathbb{R}^m} \left(\left(-\frac{m\ddot{t}_0}{2t_0} + \frac{m\dot{t}_0^2}{2t_0^2} + \frac{m^2\dot{t}_0^2}{4t_0^2} \right) + \frac{(m-2)}{2} \left(\frac{\dot{t}_0}{t_0} - \frac{m\dot{t}_0^2}{t_0^2} - \frac{2\dot{t}_0^2}{t_0^2} \right) \right) |Tf|^2 G dV \\
&\quad + \int_{\mathbb{R}^m} \left(m(m-2) \frac{\dot{t}_0^2}{4t_0^2} \right) |Tf|^2 G dV \\
&\quad + \int_{\mathbb{R}^m} \left(-\frac{2\dot{t}_0^2}{t_0} |\tau|^2 - \frac{1}{t_0} \left(|\dot{x}_0 \lrcorner Tf|^2 \right) \right) G dV.
\end{aligned}$$

We combine the non spatial dependent coefficients in the integrand of the form $\int_{\mathbb{R}^m} (\cdot) |Tf|^2 G_0 dV$ and obtain

$$\left(-\frac{\ddot{t}_0}{t_0} + \frac{2\dot{t}_0^2}{t_0^2} \right) + m \left(\frac{-\ddot{t}_0}{2t_0} + \frac{\dot{t}_0^2}{2t_0^2} + \frac{\ddot{t}_0}{2t_0} - \frac{\dot{t}_0^2}{t_0^2} + \frac{\dot{t}_0^2}{t_0^2} - \frac{\dot{t}_0^2}{2t_0^2} \right) + m^2 \left(\frac{\dot{t}_0^2}{4t_0^2} - \frac{\dot{t}_0^2}{2t_0^2} + \frac{\dot{t}_0^2}{4t_0^2} \right).$$

Thus the coefficients in front of m and m^2 vanish accordingly. Therefore we conclude that

$$\frac{t_0}{2} \int_{\mathbb{R}^m} (\dot{\mathbf{g}} + \mathbf{g}^2) |Tf|^2 G dV = \int_{\mathbb{R}^m} \left(-\frac{\ddot{t}_0}{2} + \frac{\dot{t}_0^2}{t_0} \right) |Tf|^2 G dV - \int_{\mathbb{R}^m} \left(\dot{t}_0^2 |\tau|^2 + \frac{1}{2} \left(|\dot{x}_0 \lrcorner Tf|^2 \right) \right) G dV.$$

Considering the last line of (3.17) and incorporating identities (a) and (b) of Corollary 3.9 yields

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^m} (t_0(\dot{\mathbf{g}} + \mathbf{g}^2) + \ddot{t}_0 + 2\dot{t}_0\mathbf{g}) |Tf|^2 G dV &= \int_{\mathbb{R}^m} \left(-\frac{\ddot{t}_0}{2} + \frac{\dot{t}_0^2}{t_0} + \frac{\ddot{t}_0}{2} + \dot{t}_0\mathbf{g}_0 \right) |Tf|^2 G dV \\
&\quad - \int_{\mathbb{R}^m} \left(\dot{t}_0^2 |\tau|^2 + \frac{1}{2} \left(|\dot{x}_0 \lrcorner Tf|^2 \right) \right) G dV \\
&= \int_{\mathbb{R}^m} \left(\frac{(2-m)\dot{t}_0^2}{2t_0} + \frac{\dot{t}_0^2 |x - x_0|^2}{4t_0^2} + \dot{t}_0 \frac{\langle \dot{x}_0, x - x_0 \rangle}{2t_0} \right) |Tf|^2 G dV \\
&\quad - \int_{\mathbb{R}^m} \left(\dot{t}_0^2 |\tau|^2 + \frac{1}{2} \left(|\dot{x}_0 \lrcorner Tf|^2 \right) \right) G dV \\
&= - \int_{\mathbb{R}^m} \left(\dot{t}_0^2 |\tau|^2 + \frac{1}{2} |\dot{x}_0 \lrcorner Tf|^2 \right) G dV.
\end{aligned}$$

Combining everything we conclude the result. \square

4. ENTROPY STABILITY AND \mathcal{F} -STABILITY

In this section we prove Theorem 1.1. In analogy to [3], this shows the equivalence of \mathcal{F} and entropy stability for self shrinkers provided they do not split off a line isometrically. We characterize these notions in the following definitions.

Definition 4.1. A soliton $f \in \mathfrak{S}$ is *entropy stable* if it is a local minimizer of the entropy functional λ .

Definition 4.2. A soliton $f \in \mathfrak{S}$ is called \mathcal{F} -stable if for any $X \in W_f^{2,2}$ there exists a real number $q \in \mathbb{R}$ and a constant vector field $V \in T\mathbb{R}^m$ such that $\mathcal{F}''(q, V, X) \geq 0$, (where \mathcal{F}'' is as in Proposition 3.15).

Definition 4.3. We say a map f is *cylindrical* if there is a constant vector field $\zeta \in T\mathbb{R}^m$ such that

$$\zeta \lrcorner Tf \equiv 0.$$

We build up the proof of Theorem 1.1 in a series of lemmas. First, in Lemma 4.4 we show that a soliton which is also a harmonic map is necessarily constant. Using this we show in Lemma 4.6 that the point $(0, 1) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ realizing the supremum in the definition of λ is a *strict* maximum as long as the soliton is not cylindrical. Moreover there is an effective estimate of the drop in the \mathcal{F} -functional provided the basepoint is taken a small definite distance from $(0, 1)$. Using this estimate and an observation that the \mathcal{F} -functional is strictly convex along carefully chosen paths in $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$, we obtain the proof of Theorem 1.1.

Lemma 4.4. *Suppose $f \in \mathfrak{S}$ has polynomial energy density growth and satisfies $\tau = 0$. Then f is a constant map.*

Proof. Suppose to the contrary that f is a nonconstant harmonic soliton. The corresponding self similar solution with the given f as the time -1 slice, as described in Lemma 3.4, satisfies that for all $\kappa \in \mathbb{R}$, we have $f(x, t) = f(\kappa x, \kappa^2 t)$. Thus

$$Tf_t(x) = \kappa (Tf_{\kappa^2 t}(\kappa x)),$$

Note that because f is nontrivial there exists some $y \in \mathbb{R}^m$ at which the following limit holds.

$$\lim_{t \rightarrow 0} |Tf_t(y\sqrt{-t})| = \lim_{t \rightarrow 0} \frac{1}{\sqrt{-t}} |Tf_{-1}(y)| = \infty.$$

In particular, $\sup_{\mathbb{R}^m \times [-1, 0)} |Tf_t| = \infty$.

Simultaneously since $\tau = 0$ and solutions to the (HMHF) on \mathbb{R}^m with fixed initial conditions subject to the polynomial growth condition are unique, we obtain that $\frac{\partial}{\partial t} f_t = 0$ for all $(x, t) \in \mathbb{R}^n \times [-1, 0)$. Thus for all $x \in \mathbb{R}^m$, we have $|Tf_t| = |Tf_{-1}|$. This implies that $\sup_{x \in \mathbb{R}^n, t \in [-1, 0)} |Tf_t| = \sup_{x \in \mathbb{R}^n} |Tf_{-1}| < \infty$, which is a contradiction. The result follows. \square

Definition 4.5. Given a one-parameter family of smooth maps $f_s \in C^\infty(\mathbb{R}^m, N)$, let

$$\Xi : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \times I : (x, t, s) \mapsto \mathcal{F}_{x,t}(f_s),$$

and moreover set $\Xi(x, t) := \Xi(x, t, 0)$.

Lemma 4.6. *Suppose that $f \in \mathfrak{S}$ has polynomial energy density growth and is noncylindrical. Given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that*

$$(4.1) \quad \sup_{\{(x_0, t_0) : |x_0| + |\log t_0| > \epsilon\}} \mathcal{F}_{x_0, t_0}(f) < \lambda(f) - \delta_\epsilon.$$

Proof. The argument follows closely work in [3]. The proof, which demonstrates that $(0, 1)$ is a strict global maximum, is divided into two steps. **Step 1** involves demonstrating that Ξ has a strict local maximum at $(0, 1)$, by first characterizing $(0, 1)$ as a critical point and then checking its concavity. For **Step 2** we show that Ξ decreases along a family of paths spanning over the entire space-time domain $\mathbb{R}^m \times \mathbb{R}_{\geq 0}$ which emanate from $(0, 1)$.

Step 1. Recalling the first variation formula of Proposition 3.12, since f is a soliton then by Corollary 3.13 the gradient of Ξ vanishes at $(0, 1)$, thus f is a critical point. Next if we consider

the second variation formula of Proposition 3.15 applied to a fixed f and evaluated along a path $(sy, 1 + sh)$ for $s > 0$ and $h \in \mathbb{R}$ we obtain

$$(4.2) \quad \frac{d^2}{ds^2} [\Xi(sy, 1 + sh)] = - \int_{\mathbb{R}^m} \left(h^2 |\tau_s|^2 + \frac{1}{2} |y \lrcorner T f_s|^2 \right) G_s dV.$$

Note that (4.2) is nonpositive. The first quantity on the right vanishes only if $h = 0$ or if f_s is harmonic and therefore trivial by Lemma 4.4. The second quantity on the right only vanishes if $y \lrcorner T f_s = 0$, which forces $y = 0$ since f is noncylindrical. Therefore Ξ has a strict local maximum at $(0, 1)$.

Step 2. We next show that for a given $y \in \mathbb{R}^m$ and $h \in \mathbb{R}$, one has $\frac{d}{ds} [\Xi(sy, 1 + hs^2)] \leq 0$ for all $s > 0$ such that $1 + hs^2 > 0$. Using Proposition 3.12 and replacing $x_0 \mapsto x_s$ and $t_0 \mapsto t_s$ and $G_0 \mapsto G_s$, imposing $\frac{\partial f_s}{\partial s} \equiv 0$, and then inserting identities (a) and (b) of Corollary 3.8 we obtain

$$\begin{aligned} \frac{d}{ds} [\Xi(x_s, t_s)] &= \int_{\mathbb{R}^m} \left(\dot{t}_s \left(\frac{2-m}{4} + \frac{|x - x_s|^2}{8t_s} \right) + \frac{\langle \dot{x}_s, x - x_s \rangle}{4} \right) G_s dV \\ &= \frac{\dot{t}_s}{8t_s} 2 \int_{\mathbb{R}^m} \left(|(x - x_s) \lrcorner T f_s|^2 - t_s \langle x \lrcorner T f_s, (x - x_s) \lrcorner T f_s \rangle \right) G_s dV \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^m} \langle (-x_s + x(1 - t_s)) \lrcorner T f_s, \dot{x}_s \lrcorner T f_s \rangle G_s dV. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{ds} [\Xi(x_s, t_s)] \Big|_{x_s=sy, t_s=1+hs^2} &= \frac{2hs}{8(1+hs^2)} 2 \int_{\mathbb{R}^m} \left(|(x - sy) \lrcorner T f_s|^2 - t_s \langle x \lrcorner T f_s, (x - sy) \lrcorner T f_s \rangle \right) G_s dV \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^m} \langle (sy + xhs^2) \lrcorner T f_s, y \lrcorner T f_s \rangle G_s dV \\ &= \frac{hs}{2(1+hs^2)} \int_{\mathbb{R}^m} \left(|(x - sy) \lrcorner T f_s|^2 - (1+hs^2) \langle x \lrcorner T f_s, (x - sy) \lrcorner T f_s \rangle \right) G_s dV \\ &\quad + \frac{s}{2} \frac{h}{(1+hs^2)} \int_{\mathbb{R}^m} \left\langle \frac{(1+hs^2)}{h} ((-y - xhs) \lrcorner T f_s), y \lrcorner T f_s \right\rangle G_s dV \\ &= -\frac{s}{2(1+hs^2)} \int_{\mathbb{R}^m} \left(s^2 h^2 |x \lrcorner T f_s|^2 + 2sh \langle x \lrcorner T f_s, y \lrcorner T f_s \rangle + |y \lrcorner T f_s|^2 \right) G_s dV \\ &= -\frac{s}{(1+hs^2)} \int_{\mathbb{R}^m} |(shx + y) \lrcorner T f_s|^2 G_s dV. \end{aligned}$$

Since $(1 + hs^2) \geq 0$ for all of the paths parametrized by $(sy, 1 + hs^2)$ we have that the derivative of Ξ is nonpositive on the union of these paths. We conclude the desired result. \square

Lemma 4.7. *Let $f \in C^\infty(\mathbb{R}^m)$ have polynomial growth. Then for all $x_0 \in \mathbb{R}^n$,*

$$(4.3) \quad \lim_{t_0 \rightarrow 0} \int_{\mathbb{R}^m} f G_{x_0, t_0} dV = f(x_0).$$

Furthermore given a map $f : \mathbb{R}^m \rightarrow (N, h)$ with polynomial energy density growth, then for all $x_0 \in \mathbb{R}^m$,

$$\lim_{t_0 \rightarrow 0} \mathcal{F}_{x_0, t_0}(f) = 0.$$

Proof. We have that (4.3) is a well known trait of the heat kernel. We address the second identity. Since $|Tf|^2$ has polynomial growth we have

$$\lim_{t_0 \rightarrow 0} \mathcal{F}_{x_0, t_0}(f) = \lim_{t_0 \rightarrow 0} t_0 \int_{\mathbb{R}^n} |Tf|^2 G_0 = \left(\lim_{t_0 \rightarrow 0} t_0 \right) \left(\lim_{t_0 \rightarrow 0} \int_{\mathbb{R}^n} |Tf|^2 G_0 \right) = 0,$$

as required. \square

Proof of Theorem 1.1. If f is not \mathcal{F} -stable then there exists a variation f_s for $s \in [-2\epsilon, 2\epsilon]$ with $f_0 = f$ satisfying the following

- (V1) For each variation f_s of f , the support of $f_s - f$ is compact.
- (V2) For any paths (x_s, t_s) with $x_0 = 0$ and $t_0 = 1$,

$$(4.4) \quad \left. \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(f_s)] \right|_{s=0} < 0.$$

For the one-parameter family f_s , we set Ξ to be as in Definition 4.5. Also, set

$$(4.5) \quad B_r^\circ := \{(x, t, s) : 0 < |x| + |\log t| + s < r\}.$$

We claim that there exists $\epsilon' > 0$ so that for $s \neq 0$ and $|s| \leq \epsilon'$ one has

$$(4.6) \quad \lambda(f_s) := \sup_{x_0, t_0} \Xi(x_0, t_0, s) < \Xi(0, 1, 0) = \lambda(f).$$

Following [3] we proceed in five steps:

- (1) Ξ has a strict local maximum at $(0, 1, 0)$.
- (2) $\Xi(\cdot, \cdot, 0)$ has a strict global maximum at $(0, 1, 0)$.
- (3) $\frac{\partial}{\partial s} [\Xi(x_0, t_0, s)]$ is uniformly bounded on compact sets.
- (4) For $|x_0|$ sufficiently large, $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$.
- (5) For $|\log t_0|$ sufficiently large, $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$.

These steps yield the groundwork for the desired result at the end of the proof.

Proof of (1): Since f is a soliton, by Corollary 3.13, given a path (x_s, t_s) with $(x_0, t_0) = (0, 1)$ and a variation f_s of f , we have $\frac{\partial}{\partial s} [\Xi(x_s, t_s, s)]|_{s=0} = 0$, which implies that $(0, 1, 0)$ is a critical point of Ξ . Consider one such path of the form $(sy, 1 + hs)$ for $y \in \mathbb{R}^n$, $h \in \mathbb{R}$ and some variation of f given by f_{bs} for some $b \neq 0$. Then we have that, by property (V2),

$$\left. \frac{\partial^2}{\partial s^2} [\Xi(sy, 1 + hs, bs)] \right|_{s=0} = b^2 \left. \frac{\partial^2}{\partial s^2} [\mathcal{F}_{x_s, y_s}(f_s)] \right|_{s=0} \leq 0,$$

where here $x_s = s\frac{y}{b}$ and $t_s = 1 + \frac{h}{b}s$. Now we consider the second variation when $b = 0$. As an immediate application of Proposition 4.6 we have that $\left. \frac{\partial^2}{\partial s^2} [\Xi(sy, 1 + hs, 0)] \right|_{s=0} < 0$. Therefore the Hessian of Ξ is negative definite at $(0, 1, 0)$, and thus Ξ attains a strict local maximum at this point. We may choose $\epsilon' \in (0, \epsilon)$ such that for $(x_0, t_0, s) \in B_{\epsilon'}^\circ$ we have that

$$\Xi(x_0, t_0, s) < \Xi(0, 1).$$

Proof of (2): This is an immediate result of Proposition 4.6. Therefore, $\lambda(f) = \Xi(0, 1)$ and we may choose $\delta > 0$ so that for all points of the form $(x_0, t_0, 0)$ outside $B_{\epsilon'/4}^\circ$ we have that

$$\Xi(x_0, t_0) < \Xi(0, 1) - \delta.$$

Proof of (3): Using Proposition 3.12, we see that

$$\frac{\partial}{\partial s} [\Xi(x_0, t_0, s)] = -t_s \int_{\mathbb{R}^m} \langle \dot{f}_s, S_{x_0, t_0}(f_s) \rangle G_0 dV.$$

Using the assumption of polynomial energy density growth and applying the dominated convergence theorem it follows that $\frac{\partial \Xi}{\partial s}$ is continuous in all three variables x_0, t_0 and s . Therefore $\frac{\partial \Xi}{\partial s}$ is uniformly bounded on compact sets.

Proof of (4): By hypothesis, we may choose $R \in \mathbb{R}_{>0}$ so that the support of $f - f_s$ is contained in $B_R \subset \mathbb{R}^m$. Let $\rho > 0$ and consider $|x_0| > \rho + R$. Then we have that

$$\begin{aligned} \Xi(x_0, t_0, s) &= t_0 \int_{\mathbb{R}^m} |Tf_s|^2 G_0 dV \\ &= t_0 \int_{B_R} |Tf_s|^2 G_0 dV + t_0 \int_{\mathbb{R}^m \setminus B_R} |Tf|^2 G_0 dV \\ &\leq t_0^{\frac{2-m}{2}} (4\pi)^{-\frac{m}{2}} \int_{B_R} |Tf_s|^2 e^{-\frac{|x-x_0|^2}{4t_0}} dV + \Xi(x_0, t_0, 0) \\ &\leq t_0^{\frac{2-m}{2}} (4\pi)^{-\frac{m}{2}} e^{-\frac{\rho^2}{4t_0}} \int_{B_R} |Tf_s|^2 dV + \Xi(x_0, t_0, 0). \end{aligned}$$

By compactness of the domain $B_R \times [-2\epsilon, 2\epsilon]$ we know that $\int_{B_R} |Tf_s|^2 dV < C_R$ for some $C_R \in \mathbb{R}$. Therefore we conclude that

$$(4.7) \quad \Xi(x_0, t_0, s) \leq (4\pi)^{-\frac{m}{2}} C_R t_0^{\frac{2-m}{2}} e^{-\frac{\rho^2}{4t_0}} + \Xi(x_0, t_0, 0).$$

Define the quantity

$$(4.8) \quad \mu_\rho(\tau) := \tau^{\frac{2-m}{2}} e^{-\frac{\rho^2}{4\tau}}.$$

We note in particular that

$$\mu_1\left(\frac{\tau}{\rho^2}\right) = \left(\frac{\tau}{\rho^2}\right)^{\frac{2-m}{2}} e^{-\frac{\rho^2}{4\tau}} = \rho^{m-2} \mu_\rho(\tau).$$

The function μ_1 is clearly continuous and therefore bounded and also satisfies the following limit for $\alpha \in \{0, \infty\}$,

$$\lim_{\tau \rightarrow \alpha} \mu_1(\tau) = \lim_{\tau \rightarrow \alpha} \tau^{\frac{2-m}{2}} e^{-\frac{1}{4\tau}} = 0.$$

We thus conclude that

$$(4.9) \quad \lim_{\rho \rightarrow \infty} \left(\sup_{\tau > 0} \mu_\rho(\tau) \right) = \lim_{\rho \rightarrow \infty} \sup_{\tau > 0} \left(\rho^{2-m} \mu_1\left(\frac{\tau}{\rho^2}\right) \right) = 0.$$

Therefore, as a consequence of (2) combined with this above limit, we conclude that for $|x_0|$ sufficiently large we have that $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$, as desired.

Proof of (5): We first perform the following manipulation, for $R \in \mathbb{R}_{>0}$

$$\begin{aligned} \Xi(x_0, t_0, s) &= t_0 \int_{\mathbb{R}^m} |Tf_s|^2 G_0 dV \\ &= t_0 \int_{B_R} |Tf_s|^2 G_0 dV + t_0 \int_{\mathbb{R}^m \setminus B_R} |Tf_s|^2 G_0 dV \\ (4.10) \quad &\leq t_0^{\frac{2-m}{2}} (4\pi)^{-\frac{m}{2}} \int_{B_R} |Tf_s|^2 G_0 dV + \Xi(x_0, t_0, 0) \\ &\leq C_R t_0^{\frac{2-m}{2}} (4\pi)^{-\frac{m}{2}} + \Xi(x_0, t_0, 0). \end{aligned}$$

As a result of this, we also obtain the estimate

$$\sup_{t_0 \geq 1} \Xi(x_0, t_0, s) \leq C_R (4\pi)^{-\frac{m}{2}} + \lambda(f).$$

We break into two cases. First, suppose t_0 is very large. Combining (4.10) with part (2) we obtain the claim. The case when t_0 is small, in particular $t_0 \leq 1$, is more difficult. Using Proposition 3.12

with $\dot{t}_0 = 1$, we have

$$(4.11) \quad \begin{aligned} \frac{\partial}{\partial t_0} (\Xi(x_0, t_0, s)) &= \int_{\mathbb{R}^m} \left[\frac{2-m}{4} + \frac{|x-x_0|^2}{8t_0} \right] |Tf_s|^2 G_0 dV \\ &\geq -C_0(|x_0|). \end{aligned}$$

Note that C_0 is independent of x_0 , t_0 , and s subject to the restriction $|x_0| < R$. Recall from Step (2), that

$$(4.12) \quad \Xi(0, 1, 0) = \lambda(f) > 0.$$

Choose $\alpha > 0$ so that $3\alpha < \lambda(f)$, and choose $t_\alpha = \frac{\alpha}{C_0}$. For any $x \in \mathbb{R}^m$ and $s \in [-\epsilon, \epsilon]$, by Lemma 4.7 there exists some $t_{x,s} > 0$ such that for all $t_0 \leq t_{x,s}$ we have $|\Xi(x, t_0, s)| < \alpha$.

On the set $\overline{B_{R+1}} \times [-\epsilon, \epsilon]$ we will construct a finite open cover as follows. The cover consists of balls b_i of radius $r_i > 0$ centered at (x_i, t_i) . Each b_i has an associated time $t_i \leq \min\{t_\alpha, 1\}$ where

- (1) Given (x, s) there exists an index $i(x, s)$ such that $(x, s) \in b_{i(x,s)}$.
- (2) For each b_i the associated t_i is such that

$$\Xi(x, t_i, s)|_{b_i} < \alpha.$$

Note that this choice follows from the existence of $t_{x,s}$ and the continuity of Ξ .

Choosing a finite subcover of the b_i 's we let \bar{t} be the minimum of all corresponding t_i . Then as a result of the derivative estimate 4.11 we have that for any triple (x, t_0, s) with $s \in [-\epsilon, \epsilon]$, and $x \in \overline{B_R}$, and $t_0 \leq \bar{t}$,

$$\Xi(x, t_0, s) \leq \Xi(x, t_{i(x,s)}, s) + C_0(t_{i(x,s)} - t_0) \leq 2\alpha < \lambda(f).$$

Claim (5) follows.

Given claims (1)-(5) we finish the proof by dividing the domain into regions corresponding to the size of $|x_0| + |\log t_0|$. Using (1), when s is sufficiently small there exists some $r > 0$ such that $\Xi(x_0, t_0, s) < \Xi(x_0, t_0, 0)$ for (x_0, t_0) within the following region

$$\mathfrak{R}_1 := \{(x_0, t_0) : |x_0| + |\log t_0| < r\}.$$

Using (4) and (5) there exists an $R > 0$ such that $\Xi(x_0, t_0, s) < \Xi(x_0, t_0, 0)$ for (x_0, t_0) in the following region.

$$\mathfrak{R}_2 := \{(x_0, t_0) : |x_0| + |\log t_0| > R\}.$$

Therefore it remains to consider

$$(4.13) \quad \mathfrak{R}_3 := \{(x_0, t_0) : R > |x_0| + |\log t_0| > r\}.$$

Given $(x_0, t_0) \in \mathfrak{R}_3$, we know by (2) that $\Xi(x_0, t_0, 0) < \lambda(f)$, and by (3) that the s derivative of Ξ is uniformly bounded. So we may choose a $\delta > 0$ such that Ξ restricted to the region $\mathfrak{R}_3 \times [-\delta, \delta]$ is bounded above by $\lambda(f)$. Therefore, (4.6) holds on $\bigcup_{i=1}^3 \mathfrak{R}_i$ and as this union constitutes the entire space-time domain, the result follows.

Therefore we conclude that entropy stability implies \mathcal{F} -stability. For the converse, assume that f is \mathcal{F} -stable. Then by definition for any variation f_s of f there exists some $\sigma \in \mathbb{R}$ and $\zeta \in TM$ such that $\mathcal{F}''(\dot{f}_0, \sigma, \zeta) \geq 0$. Hence for families x_s, t_s with $\dot{x}_s = \zeta$ and $\dot{t}_s = \sigma$ and sufficiently small s we have

$$\begin{aligned} \lambda(f_s) &\geq \mathcal{F}_{x_s, t_s}(f_s) \\ &\geq \mathcal{F}_{0,1}(f) \\ &= \lambda(f). \end{aligned}$$

Therefore we have that $\lambda(f_s) \geq \lambda(f)$ for any variation f_s of f , and the result follows. \square

5. RIGIDITY RESULTS

In this section we establish rigidity results for solitons, focusing primarily on the case of spherical targets. Our main goal is to establish Theorem 1.3. To accomplish this we first show Theorem 1.2, which shows that showing a negative upper bound for the Rayleigh quotient characterizing the lowest eigenmode of L^f suffices to establish instability of a given soliton. With this in place, by carefully exploiting conformal vector fields on the sphere we establish such an upper bound in the presence of large entropy, finishing the proof of Theorem 1.3.

5.1. \mathcal{F} -stability and Eigenvector fields of L^f . For the rest of the paper we only consider solitons based at $(0, 1)$, and drop the basepoint on the Greens function in our notation. Given the equivalence of entropy stability and \mathcal{F} -stability, we will study spectral properties of the L^f operator, which for a soliton based at $(0, 1)$ is

$$L^f V = -\Delta V - R(V, T f_\alpha) T f_\alpha + \nabla_{\frac{x}{2}} V.$$

Our first lemma concerns the action of L^f on the pushforward of a vector field.

Lemma 5.1. *Suppose $f \in C^\infty(\mathbb{R}^m, N)$ is a soliton. Given a vector field $\zeta^\alpha \partial_\alpha \in T\mathbb{R}^m$,*

$$L^f (\zeta \lrcorner T f) = - \left(\frac{\zeta}{2} \lrcorner T f \right) - 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha \partial_\beta f) - T f(\Delta \zeta) + \left(\nabla_{\frac{x}{2}} \zeta \right) \lrcorner T f.$$

Proof. Recall that by the definition of L^f ,

$$L^f (\zeta \lrcorner T f)^j = -\nabla_\beta \nabla_\beta [\zeta^\gamma T f_\gamma^j] - (\partial_\alpha f^q)(\partial_\alpha f^j)(f^p) R_{pqj}^i \zeta^\eta T f_\eta^i - \frac{x^\alpha}{2} \nabla_\alpha (\zeta^\eta T f_\eta^j).$$

We first compute

$$\begin{aligned} \Delta (\zeta^\beta (f_\beta^i)) &= \nabla_\alpha \nabla_\alpha (\zeta^\beta f_\beta^i) \\ &= \nabla_\alpha [\zeta^\beta \nabla_\alpha f_\beta^i] + \nabla_\alpha (\nabla_\alpha \zeta^\beta f_\beta^i) \\ &= \zeta^\beta (\nabla_\alpha \nabla_\alpha f_\beta^i) + 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha f_\beta^i) + (\nabla_\alpha \nabla_\alpha \zeta^\beta) f_\beta^i. \end{aligned}$$

We expand the first term:

$$\begin{aligned} \zeta^\beta (\nabla_\alpha \nabla_\alpha (f_\beta^i)) &= \zeta^\beta (\nabla_\alpha \nabla_\beta (f_\alpha^i)) \\ &= \zeta^\beta ([\nabla_\alpha, \nabla_\beta] f_\alpha^i) + \zeta^\beta (\nabla_\beta \nabla_\alpha f_\alpha^i) \\ &= \zeta^\beta ((\partial_\alpha f^p)(\partial_\beta f^q) R_{pqk}^i f_\alpha^k) + \zeta^\beta (\nabla_\beta \nabla_\alpha f_\alpha^i). \end{aligned}$$

We apply the soliton equation to the first term

$$\begin{aligned} \zeta^\beta \nabla_\beta \tau^i &= \zeta^\beta \nabla_\beta \left(\frac{x^\alpha}{2} \partial_\alpha f^i \right) \\ &= \frac{\zeta^\alpha}{2} (\partial_\alpha f^i) + \zeta^\beta \frac{x^\alpha}{2} \nabla_\beta (\partial_\alpha f^i) \\ &= \frac{\zeta^\alpha}{2} (\partial_\alpha f^i) + \zeta^\beta \frac{x^\alpha}{2} \nabla_\beta (\partial_\alpha f^i). \end{aligned}$$

Therefore

$$\begin{aligned} \Delta (\zeta^\beta (\partial_\beta f^i)) &= \zeta^\beta ((\partial_\alpha f^p)(\partial_\beta f^q) R_{pqk}^i (\partial_\alpha f^k)) + \frac{\zeta^\alpha}{2} (\partial_\alpha f^i) \\ &\quad + \zeta^\beta \frac{x^\alpha}{2} \nabla_\beta (\partial_\alpha f^i) + 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha \partial_\beta f^i) + (\nabla_\alpha \nabla_\alpha \zeta^\beta)(\partial_\beta f^i). \end{aligned}$$

Therefore

$$\begin{aligned} L^f(\zeta \lrcorner Tf)^i &= -\frac{1}{2}\zeta^\alpha(\partial_\alpha f^i) - \zeta^\beta \frac{x^\alpha}{2} \nabla_\beta(\partial_\alpha f^i) - 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha \partial_\beta f^i) - (\nabla_\alpha \nabla_\alpha \zeta^\beta)(\partial_\beta f^i) + \frac{x^\beta}{2} \nabla_\beta [\zeta^\alpha \partial_\alpha f^i] \\ &= -\frac{1}{2}\zeta^\alpha(\partial_\alpha f^i) - 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha \partial_\beta f^i) - (\nabla_\alpha \nabla_\alpha \zeta^\beta)(\partial_\beta f^i) + \frac{x^\beta}{2}(\nabla_\beta \zeta^\alpha)(\partial_\alpha f^i). \end{aligned}$$

Thus we conclude that

$$L^f(\zeta \lrcorner Tf)^i = -\frac{1}{2}\zeta^\alpha(\partial_\alpha f^i) - 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha \partial_\beta f^i) - (\nabla_\alpha \nabla_\alpha \zeta^\beta)(\partial_\beta f^i) + \frac{x^\beta}{2}(\nabla_\beta \zeta^\alpha)(\partial_\alpha f^i).$$

The result follows. \square

Corollary 5.2 ([16] Proposition 6). *Suppose $\zeta^i \partial_i$ is a constant vector field. Then we have that*

$$L^f(\zeta \lrcorner Tf) = -\frac{1}{2}\zeta \lrcorner Tf.$$

Furthermore,

$$L^f\left(\frac{x}{2} \lrcorner Tf\right) = -\frac{x}{2} \lrcorner Tf.$$

Corollary 5.3. *The following equality holds*

$$\left\langle L^f(\zeta \lrcorner Tf), (\zeta \lrcorner Tf) \right\rangle = -\frac{1}{2}|\zeta \lrcorner Tf|^2 - \left\langle \left(\Delta \zeta - \left(\nabla_{\frac{x}{2}} \zeta \right) \right) \lrcorner Tf + 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha Tf_\beta), \zeta \lrcorner Tf \right\rangle.$$

Proof. We simply compute using the formula of Lemma 5.1.

$$\begin{aligned} \left\langle L^f(\zeta \lrcorner Tf), (\zeta \lrcorner Tf) \right\rangle &= -\frac{1}{2}|\zeta \lrcorner Tf|^2 - 2\nabla_\alpha \zeta^\beta \langle \nabla_\alpha Tf_\beta, \zeta \lrcorner Tf \rangle - \langle (\Delta \zeta) \lrcorner Tf, \zeta \lrcorner Tf \rangle \\ &\quad + \left\langle \left(\nabla_{\frac{x}{2}} \zeta \right) \lrcorner Tf, \zeta \lrcorner Tf \right\rangle \\ &= -\frac{1}{2}|\zeta \lrcorner Tf|^2 - \left\langle (\Delta \zeta) \lrcorner Tf + 2\nabla_\alpha \zeta^\beta \nabla_\alpha Tf_\beta - Tf \left(\nabla_{\frac{x}{2}} \zeta \right), \zeta \lrcorner Tf \right\rangle \\ &= -\frac{1}{2}|\zeta \lrcorner Tf|^2 - \left\langle \left(\Delta \zeta - \left(\nabla_{\frac{x}{2}} \zeta \right) \right) \lrcorner Tf + 2(\nabla_\alpha \zeta^\beta)(\nabla_\alpha Tf_\beta), \zeta \lrcorner Tf \right\rangle. \end{aligned}$$

The result follows. \square

Next we give a generalization of ([16] Theorem 3).

Proposition 5.4. *Suppose $f : \mathbb{R}^m \rightarrow N$ is a soliton and the sectional curvature of N is bounded above by $K \in \mathbb{R}_{>0}$. If $K|Tf|^2 \leq 1$ then f is constant. In particular, if the sectional curvature of N is nonpositive then any soliton whose target is N must be constant.*

Proof. Integrating by parts, we get

$$\begin{aligned}
\int_{\mathbb{R}^m} |\nabla \tau|^2 G_0 dV &= \int_{\mathbb{R}^m} \langle \nabla_\alpha \tau, \nabla_\alpha \tau \rangle G_0 dV \\
&= - \int_{\mathbb{R}^m} \langle \tau, \Delta \tau \rangle G_0 dV + \int_{\mathbb{R}^m} \left\langle \frac{x}{2} \lrcorner \nabla \tau, \tau \right\rangle G_0 dV \\
&= \int_{\mathbb{R}^m} \left\langle -\Delta \tau + \frac{x}{2} \lrcorner \nabla \tau, \tau \right\rangle G_0 dV \\
&= \int_{\mathbb{R}^m} \left\langle L^f \tau + R(\tau, T f_\alpha) T f_\alpha, \tau \right\rangle G_0 dV \\
&= \int_{\mathbb{R}^m} \langle -\tau + R(\tau, T f_\alpha) T f_\alpha, \tau \rangle G_0 dV \\
&= \int_{\mathbb{R}^m} (-|\tau|^2 + \langle R(\tau, T f_\alpha) T f_\alpha, \tau \rangle) G_0 dV \\
&\leq \int_{\mathbb{R}^m} |\tau|^2 (K|T f|^2 - 1) G_0 dV.
\end{aligned}$$

Therefore τ is parallel and $|\tau|$ is constant. By the soliton equation and using polar coordinates,

$$\tau = \left(\frac{x}{2} \lrcorner T f \right) = \frac{1}{2} r \partial_r f.$$

Therefore $|\tau|^2 = \frac{r^2}{4} [\partial_r f]^2 \in \mathbb{R}$. Since this is true for all r , we must have that $\partial_r f = \tau = 0$. Thus f is a constant map. \square

Definition 5.5. Let $f : \mathbb{R}^m \rightarrow N$ be a soliton. The bottom of the spectrum of L^f is defined to be

$$\mu_1 = \inf_V \frac{\int_{\mathbb{R}^m} \langle L^f V, V \rangle G_0 dV}{\int_{\mathbb{R}^m} |V|^2 G_0 dV},$$

taken over smooth vector fields $V \in C_0^\infty(f^*TN)$ of compact support.

We note that this infimum can be taken over Lipschitz vector fields V in the space $W_f^{2,2}$ as defined in 3.16. At first glance there is nothing stopping this infimum from being $-\infty$, but Theorem 1.2, which we now prove, shows that if μ_1 is sufficiently negative, then the soliton in question is unstable.

Proof of Theorem 1.2. We follow closely the corresponding Lemma 9.58 in [3]. Let $\mu_1(R)$ denote the smallest Dirichlet eigenvalue of L^f in a ball of radius R centered at the origin. Observe that $\mu_1(R)$ is decreasing and $\lim_{R \rightarrow \infty} \mu_1(R) = \mu$. Choose some $R > 0$ big enough such that $\mu_1(R) < -\frac{3}{2}$ and let V be a nonzero Dirichlet eigenfield $L^f V = \mu_1(R)V$ on the ball B_R . Let $h \in \mathbb{R}$, $y \in \mathbb{R}^m$, and consider the second variation identity for \mathcal{F} in the directions determined by V , h and y as in 3.15, giving

$$\mathcal{F}''(h, y, V) = \int_{B_R} \left(\langle V, L^f V - 2h\tau - (y \lrcorner T f) \rangle - h^2 |\tau|^2 - \frac{1}{2} |y \lrcorner T f|^2 \right) G_0 dV.$$

We apply Young's inequality to the term $-\langle V, y \lrcorner T f \rangle \leq \frac{1}{2}(|V|^2 + |y \lrcorner T f|^2)$ and find that this is bounded above by

$$\int_{B_R} \left(\left(\frac{1}{2} + \mu_1(R) \right) |V|^2 - 2h \langle V, \tau \rangle - t^2 |\tau|^2 \right) G_0 dV \leq \int_{B_R} -|V|^2 - 2h (\langle V, \tau \rangle - h^2 |\tau|^2) G_0 dV < 0,$$

for all choices of h and y . It follows that f is \mathcal{F} -unstable. \square

We note that if one could prove the result that if $\mu_1 > -\infty$ then there is a vector field $V \in W_f^{2,2}$ such that $L^f V = \mu_1 V$, a modification of the previous proof could show that $\mu_1 < -1$ implies \mathcal{F} instability.

5.2. Solitons with Spherical Targets. In this section we will restrict our discussion to solitons $f : \mathbb{R}^m \rightarrow S^n$ with spherical target. In fact, in light of Proposition 5.4, there are no nontrivial solitons without some positive sectional curvature on the target. Our restriction is therefore natural because the sphere is the simplest manifold for which a finite time singularity in the harmonic map heat flow can occur.

An interesting observation about the harmonic map heat flow from a compact manifold into the sphere S^n , $n \geq 3$, is that any harmonic map $f : M \rightarrow S^n$ is unstable or constant. This is because the second variation of the energy at such a harmonic map satisfies

$$\frac{d^2}{dt^2} [\mathcal{E}(f_t)] = \int_{\mathbb{R}^m} |\nabla V|^2 - |Tf|^2 |V|^2 + |\langle V, Tf \rangle|^2 dV,$$

which, remarkably, is negative when summed over an appropriate basis of conformal vector fields. Through a similar calculation, we will study the \mathcal{F} -stability of a soliton $f : \mathbb{R}^m \rightarrow S^n$ and its relationship to conformal vector fields. Recall that a conformal vector field W on the sphere S^n is constructed by fixing some $w \in \mathbb{R}^{n+1}$ and considering the vector field whose value at $p \in S^n$ is

$$W_p := w - \langle w, p \rangle p,$$

the projection of the constant vector field w to the tangent space at p . We record a preliminary lemma on these vector fields.

Lemma 5.6. *Let w_0, w_1, \dots, w_n be any orthonormal basis of \mathbb{R}^{n+1} . Let W_i be the corresponding conformal vector fields on S^n and let v be any vector tangent to the sphere at some point p . Then*

- (1) $\sum_{i=0}^n |W_i|^2 = n$,
- (2) $\sum_{i=0}^n \langle W_i, v \rangle^2 = |v|^2$.

Proof. Given an arbitrary $p \in S^n$, using that $W = w - \langle w, p \rangle p$ we have

$$\begin{aligned} |W_i|^2 &= |w_i|^2 - 2\langle w_i, p \rangle^2 + \langle w, p \rangle^2 |p|^2 \\ &= 1 - \langle w, p \rangle^2. \end{aligned}$$

Hence

$$\sum_{i=0}^n |W_i|^2 = (n+1) - \sum \langle w_i, p \rangle^2 = n+1 - |p|^2 = n,$$

giving the first claim. Then second claim is a similar calculation, noting that if v is in the tangent space at p then

$$\langle W_i, v \rangle = \langle w_i - \langle w_i, p \rangle p, v \rangle = \langle w_i, v \rangle.$$

The result follows. □

Lemma 5.7. *Let W be a conformal vector field on the sphere corresponding to a choice $w \in \mathbb{R}^{n+1}$, and let $\omega(p) := \langle w, p \rangle$. Then for any $f : M \rightarrow S^n$ we have*

- (1) $\nabla_\alpha W = -\omega T f_\alpha$.
- (2) $\Delta W = -\langle W, T f_\alpha \rangle T f_\alpha - \omega \tau$.

Proof. Let $\overset{\circ}{\nabla}$ denote the Levi-Civita connection on S^n , and $\overline{\nabla}$ denote the Levi-Civita connection on \mathbb{R}^{n+1} . The undecorated connection will be the induced connection on $TM \otimes f^*(TN)$. Then we have that $\overset{\circ}{\nabla}(\cdot) = (\overline{\nabla}(\cdot))^T$, where the T superscript indicates projection onto the tangent space of

the sphere. We compute

$$\begin{aligned}
\nabla_\alpha W &= f_\alpha^i \overset{\circ}{\nabla}_i W \\
&= f_\alpha^i (\bar{\nabla}_i W)^T \\
&= f_\alpha^i \left(\bar{\nabla}_i [V^j \partial_j - V_k x^k x^j \partial_j] \right)^T \\
&= f_\alpha^i \left((-V_k \delta_i^k x^j - V_k x^k \delta_i^j) \partial_j \right)^T \\
&= f_\alpha^i \left(-V_i x^j \partial_j - V_k x^k \partial_i \right)^T \\
&= f_\alpha^i \left(-V_k x^k \partial_i \right) \\
&= -\omega f_\alpha^i \partial_i.
\end{aligned}$$

This proves the first claim. Next we have that

$$\begin{aligned}
\nabla_\alpha \nabla_\alpha W &= -\nabla_\alpha [\omega T f_\alpha] \\
&= -(\nabla_\alpha \omega) T f_\alpha - \omega (\nabla_\alpha T f_\alpha) \\
&= -((\partial_\alpha f^i) \overset{\circ}{\nabla}_i \omega) T f_\alpha - \omega \tau \\
&= -(\partial_\alpha f^i) (\bar{\nabla}_i (V^k x^k))^T T f_\alpha - \omega \tau \\
&= -(\partial_\alpha f^i) (V_i)^T T f_\alpha - \omega \tau \\
&= -(\partial_\alpha f^i) W_i T f_\alpha - \omega \tau \\
&= -\langle W, T f_\alpha \rangle T f_\alpha - \omega \tau.
\end{aligned}$$

The result follows. \square

Proposition 5.8. *Let W be a conformal vector field on the sphere S^n and let $f : \mathbb{R}^m \rightarrow S^n$ be a soliton. Then*

$$L^f W = 2\langle W, T f_\alpha \rangle T f_\alpha - |T f|^2 W,$$

and consequently

$$\int_{\mathbb{R}^m} \langle L^f W, W \rangle G_0 dV = \int_{\mathbb{R}^m} -|T f|^2 |W|^2 + 2|\langle W, T f \rangle|^2 G_0 dV.$$

Proof. Recall that $L^f W = -\Delta W - R(W, T f_\alpha) T f_\alpha + \frac{1}{2} \nabla_x W$. Using Lemma 5.7 we have

$$-\Delta W = \langle W, T f_\alpha \rangle T f_\alpha + \omega \tau.$$

Next, using the well-known formula for the curvature of the sphere we have

$$-R(W, T f_\alpha) T f_\alpha = -|T f|^2 W + \langle W, T f_\alpha \rangle T f_\alpha.$$

Lastly, using Lemma 5.7 again yields $\frac{x^\alpha}{2} \nabla_\alpha W = -\omega \left(\frac{x}{2} \right) \lrcorner T f$. But by the soliton identity this is $-\omega \tau$. Adding these gives the result. \square

Corollary 5.9. *Let $f : \mathbb{R}^m \rightarrow S^n$ be a soliton. If there is a conformal vector field which is perpendicular to the image of $T f$ at each point then f is either unstable or constant.*

Proof. Let W be such a vector field. Then by Proposition 5.8 we have

$$\int_{\mathbb{R}^m} \langle L^f W, W \rangle G_0 dV = - \int_{\mathbb{R}^m} |W|^2 |T f|^2 G_0 dV \leq 0,$$

since by assumption $\langle W, T f \rangle = 0$. Now if V is the pushforward of any vector field from \mathbb{R}^m , then $\langle V, W \rangle = 0$. In particular, W is orthogonal to the eigenspaces associated to the pushforward of constant vector fields and the pushforward of the position vector field. If f is stable these are

the only negative eigendirections, but on the other hand if f is not constant we would have strict negativity of $\int \langle L^f W, W \rangle G_0 dV$. \square

Corollary 5.10. *Let w_i , $0 \leq i \leq n$ be an orthonormal basis of \mathbb{R}^{n+1} and let W_i be the corresponding conformal vector fields on S^n . Then*

$$\sum_{i=0}^n \int_{\mathbb{R}^m} \langle L^f W_i, W_i \rangle G_0 dV = (2-n) \int_{\mathbb{R}^m} |Tf|^2 G_0 dV.$$

Proof. This follows by combining Lemma 5.6 and Proposition 5.8. \square

We see from this last corollary that good lower bounds on the spectrum of L^f , which as we have seen is intimately related to the notion of \mathcal{F} -stability, gives an upper bound on the entropy $\int |Tf|^2 G_0 dV$. Specifically, we are now equipped to prove the rough upper bound on the entropy given in Theorem 1.3.

Proof of Theorem 1.3. Let $f : \mathbb{R}^m \rightarrow S^n$, $n \geq 3$, be a stable soliton. Let W_i be any basis of conformal vector fields as before. By Theorem 1.2 we have $\int \langle L W_i, W_i \rangle G_0 dV \geq -\frac{3}{2} \int |W_i|^2 G_0 dV$, while by 5.10 we get

$$-\frac{3n}{2} = -\frac{3}{2} \int_{\mathbb{R}^m} \sum_{i=0}^n |W_i|^2 G_0 dV \leq (2-n) \int_{\mathbb{R}^m} |Tf|^2 G_0 dV.$$

The result follows. \square

Remark 5.11. It is shown in [14], that small entropy suffices to establish regularity of harmonic map heat flow. This result shows that, for maps into spheres, stable singularities cannot have entropy which is too large.

It is an open problem to determine exactly when a homotopy class of maps $f : S^m \rightarrow S^n$, $m \geq 3$ contains a harmonic representative. When $m = n \leq 7$ or $m = n = 9$ this problem is answered in the affirmative through a harmonic join construction, as explained in Chapter 9 of [4]. It may be the case that for sufficiently high degree maps from $S^n \rightarrow S^n$, the entropy as well as the entropy of any blowup model, is large. If so Theorem 1.3 would imply that all such singularities are unstable, and one could then use a generic harmonic map heat flow to construct harmonic maps in these homotopy classes.

6. DIMENSIONAL AUGMENTATION AND STABILITY

The idea of entropy stability aims to allow perturbations of a given map to move past all but the “essential,” stable singularities of a given harmonic map heat flow. In this section we extend this strategy by allowing a further embedding into a higher dimensional image before perturbing. As we will see this is particularly well-suited to the case of the sphere, and motivates certain conjectures on the structure of solitons mapping into spheres.

6.1. Convex domains and harmonic map flow. In this subsection we record some notions of convexity and discuss their relationship to harmonic maps, and the singularity formation of harmonic map flow, in preparation for Theorem 1.4.

Definition 6.1. Let (M, g) be a smooth Riemannian manifold. Given any open set $U \subset M$, a function $F \in C^2(U)$ is *convex* if the Hessian of F is nonnegative on U . This convexity is *strict* if nonnegativity is replaced with strict positivity. Given $(M, \partial M, g)$ a manifold with boundary, we say that a function $F : M \rightarrow \mathbb{R}$ is *boundary defining* if it is strictly convex on M and there exists $c \in \mathbb{R}$ such that $M = F^{-1}(-\infty, c]$ and $\partial M = F^{-1}(c)$.

Definition 6.2. A set $U \subset M$ is *convex supporting* if for all $K \subset U$ compact, K admits a strictly convex function F_K . Note that these functions do not have to be defined on all of U . A *maximal open convex supporting set* is one which is not properly contained in any other open convex supporting set.

It is important to note that the notion of a convex supporting set is strictly more general than that of a domain admitting a convex function. An example of such a maximal open convex supporting set is constructed in ([10], pp.5-9), and we review this construction here for convenience.

Proposition 6.3 (§2.2 and Theorem 2.1 of [10], pp.6-9). *Given $(S^n, g) \subset \mathbb{R}^{n+1}$, let*

$$H_+^n := \{(x_i)_{i=1}^{n+1} \in S^n : x_1 = 0, x_2 \geq 0\}.$$

Then $S^n \setminus H_+^n$ is a maximal open convex supporting subset of S^n .

Sketch of proof. First we show that $S^n \setminus H_+^n$ is a convex supporting subset of S^n , and address the maximality later. We will take an arbitrary compact set $K \subset (S^n \setminus H_+^n)$ and displaying a convex function over K . To do so consider the projection

$$\begin{aligned} \pi : S^n &\rightarrow \overline{D^2} \\ &: (x_i)_{i=1}^{n+1} \mapsto (x_1, x_2). \end{aligned}$$

Where D^2 is the 2-disk and the overline indicates taking its closure. Observe that by construction, $x \in (S^n \setminus H_+^n)$ if and only if $\pi(x)$ is contained in D^2 with the radius connecting the origin to $(0, 1)$. We use this projection and consider the families of equivalence classes in terms of spherical coordinates, with

$$\pi(x) = (\nu_x \sin \varphi_x, \nu_x \cos \varphi_x).$$

Over K , we have that there is some constant $c \in (0, 1)$ so that $c > \nu_x$ for all $x \in K$. Then if we choose $\kappa \in \mathbb{R}_{>0}$ sufficiently large, the function

$$F_K(x) := \kappa^{-1} \exp \left[\kappa \left(\varphi_x + \arcsin \left(\frac{c}{\nu_x} \right) \right) \right].$$

is strictly convex over K .

Now we show that $S^n \setminus H_+^n$ is maximal. Assume that there is some larger open convex supporting set, W^n containing $S^n \setminus H_+^n$. This implies that there is some $\theta \in (0, \frac{\pi}{2}]$ so that the point $(0, \sin \theta, y \cos \theta) \in S^1 \times S^{n-1}$ resides in W^n . Observe that the closed geodesic

$$\begin{aligned} \gamma_t : \mathbb{R} &\rightarrow S^1 \times S^{n-1} \\ \gamma_t &:= (\sin t, \cos t \sin \theta, y \cos t \cos \theta) \end{aligned}$$

lies in W^n and has unit speed. We can take an open neighborhood U about $\gamma(\mathbb{R})$ and construct a strictly convex function F_U over U as above.

$$\frac{d^2}{dt^2} [F_U \circ \gamma_t] = \nabla^2 F_U \left(\frac{\partial \gamma_t}{\partial t}, \frac{\partial \gamma_t}{\partial t} \right) > 0.$$

But since $(F_U \circ \gamma_t)$ is periodic it achieves its maximum, a contradiction. \square

By an elementary argument, Gordon shows [6] that smooth harmonic maps into convex supporting domains are constant. We adapt this argument to show that solitons mapping into convex supporting domains are also constant.

Proposition 6.4. *Any soliton mapping into a convex supporting domain is constant.*

Proof. We give two proofs. On the one hand, a soliton satisfies the requirements of an n -obstruction. But the proof of Theorem 6.9 shows that n -obstructions do not exist. On the other hand we can give a direct proof inspired by the entropy functional. A direct calculation shows that if F is the given convex function, one has

$$\Delta(F \circ f) \geq C|Tf|^2 + \left\langle \nabla F, \frac{x}{2} \cdot Tf \right\rangle.$$

Now fix some $R > 0$, let η denote a cutoff function for B_R and then integrate by parts against a Greens function, and estimate with the Cauchy-Schwarz inequality to yield

$$\begin{aligned} \int_{B_R} |Tf|^2 \eta^2 G dV &\leq \int_{B_R} \left[\Delta(F \circ f) - \left\langle \nabla F, \frac{x}{2} \cdot Tf \right\rangle \right] \eta G dV \\ &= \int_{B_R} \langle \nabla(F \circ f), \nabla \eta \rangle \eta G dV \\ &\leq C \int_{B_R \setminus B_{\frac{R}{2}}} |Tf| |\nabla \eta| \eta G dV \\ &\leq C\epsilon \int_{B_R \setminus B_{\frac{R}{2}}} |Tf|^2 \eta^2 G dV + \frac{C}{\epsilon} \int_{B_R \setminus B_{\frac{R}{2}}} |\nabla \eta|^2 G dV \\ &\leq \frac{1}{2} \int_{B_R} |Tf|^2 \eta^2 G dV + CR^{n-2} e^{-R^2}. \end{aligned}$$

Rearranging this and choosing R sufficiently large shows that $Tf = 0$, as required. \square

Given this, a natural question to ask is if the harmonic map flow admits a smooth long time solution with convergence to a point given an initial mapping into a convex supporting region. Our next proposition shows that this is not the case, and is the reason we are forced to make stronger convexity hypotheses in showing Theorem 1.4.

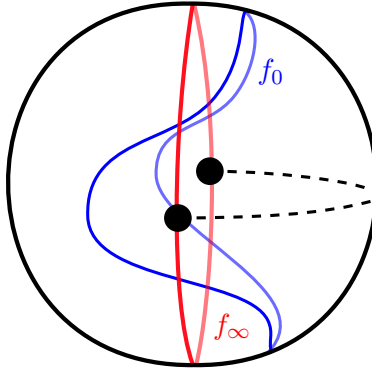


FIGURE 1. Harmonic map heat flow on sphere which exits the region $S^2 \setminus H_+^2$

Proposition 6.5. *There exists a map $f : S^1 \rightarrow S^2 \setminus H_+^2$ such that the harmonic map heat flow with initial condition f exists smoothly for all time and converges to the great circle $\{x_3 = 0\}$. Therefore this solution will leave, in finite time, the domain of definition of any convex function defined on the image of f .*

Proof. Figure 1 summarizes the situation. Let $x_1 := \cos \theta \sin \phi$, $x_2 := \sin \theta \sin \phi$, and $x_3 := \cos \phi$ be spherical coordinates. Consider the hinge H_+^2 where $x_2 = 0$ and $x_3 \geq 0$. Let $\epsilon > 0$ and consider the closed curve in $S^2 \setminus H_+^2$ given by

$$f(s) = (\theta(s), \phi(s)) = \left(s, \frac{\pi}{2} + \epsilon \cos(2s)\right).$$

This curve is invariant (up to unit speed re-parameterization) under the isometry given by reflection in the $x_3 = 0$ plane followed by a rotation of $\frac{\pi}{2}$ radians around the x_3 -axis. By uniqueness of solutions to the flow, this symmetry is preserved along the solution to the flow beginning from f . Since the domain has dimension 1, we know that the flow exists for all time and converges to a closed geodesic. As the symmetry is preserved in the limit, the limit must be the great circle with $\{x_3 = 0\}$. \square

Remark 6.6. It seems quite likely that by allowing initial data which “bends” further around the set H_+^2 that the flow can exit the entire convex supporting domain in finite time, instead of infinite time.

6.2. Proof of Theorem 1.4. In this subsection we prove Theorem 1.4, which relies strongly on results from [11]. To begin we provide a sketch of the discussion in §5.4 of ([11], pp.118-123) which discusses a generalization of the classic theorem of Eells-Sampson [5], where the nonpositivity assumption on the sectional curvature of N is replaced with the weaker assumption that the universal cover of the target N admits a strictly convex function which grows quadratically. We include sketches of some proofs for convenience as we will reference and exploit some of the arguments as well as the results themselves. First we give a preliminary definition, which will ultimately be key in tying in this work with that of the discussion of solitons.

Definition 6.7. A smooth map $v_t \in C^\infty(\mathbb{R}^n \times \mathbb{R}_{\leq 0}, N)$ is an n -obstruction for (N, h) if it satisfies the following

- (nO1) v_t satisfies (HMHF),
- (nO2) $|(Tv_0)(0)| > 0$, and for all $(x, t) \in \mathbb{R}^m \times \mathbb{R}_{\leq 0}$.

$$|(Tv_t)(x)| \leq |(Tv_0)(0)|,$$

- (nO3) There exists some $E_0 \in \mathbb{R}_{>0}$ such that for all $R > 0$ and $t \in \mathbb{R}_{\leq 0}$,

$$(6.1) \quad R^{2-m} \int_{B_R} |Tv_t|^2 \leq E_0.$$

The next result shows that n -obstructions are the obstruction to smooth existence of the harmonic map heat flow.

Proposition 6.8 ([11] Proposition 5.4.2). *Assume (N, h) does not admit an n -obstruction. Given initial data (HMHF) has a unique smooth solution $f_t \in C^\infty(M \times \mathbb{R}_{\geq 0}, N)$. Moreover, there exists $C \in \mathbb{R}_{\geq 0}$ such that for all $t \geq 1$,*

$$(6.2) \quad \|Tf_t\|_{C^0(M)} \leq C.$$

Sketch of proof. The proof consists of constructing a blowup limit around a potential singularity and showing it is an n -obstruction. If the singularity occurs for time $T < \infty$, one constructs an n -obstruction from the blowup function obtained by dilating and shifting coordinates, with some sequence of scalars $\kappa_i \searrow 0$

$$f_t^i(x) := f_{t_0 + \kappa_i^2 t}(x_0 + \kappa_i x).$$

This immediately satisfies (nO1), and higher order regularity of parabolic equations yields (nO2). Property (nO3) is verified by comparing the left side of (6.1) to a scaled version of $\mathcal{F}_{x_0, t_0}(f_t)$ and then using the monotonicity given by Lemma 2.7. In the case of a singularity at time infinity, we can choose a sequence of times $t_j \nearrow \infty$ and repeat the argument above. \square

Theorem 6.9 (Theorem 5.4.3 of [11], pp.120). *Let (\tilde{N}, \tilde{h}) be the universal covering of (N, h) . Suppose that \tilde{N} admits a strictly convex function $F \in C^2(\tilde{N})$ with quadratic growth, i.e. there are*

positive $c_0, c_1, c_2 \in \mathbb{R}_{\geq 0}$ such that

$$(6.3) \quad \nabla^2 F \geq c_0 \tilde{h} \text{ on } \tilde{N},$$

$$(6.4) \quad 0 \leq F(y) \leq c_1 (d_{\tilde{h}}(y, y_0))^2 + c_2 \text{ for all } y \in \tilde{N},$$

for some $y_0 \in \tilde{N}$, where $d_{\tilde{h}}$ is the distance function on (\tilde{N}, \tilde{h}) . Then given $f : (M^m, g) \rightarrow (N^n, h)$ there exists a smooth solution $f_t \in C^\infty(M \times \mathbb{R}_{\geq 0}, N)$ to (HMHF) satisfying (6.2). And for suitable $t \nearrow \infty$, f_t converges to a smooth harmonic map $f_\infty : M \rightarrow N$ in $C^2(M, N)$.

Sketch of proof. It amounts to appealing to Proposition 6.8 by demonstrating that (N, h) does not admit any n -obstructions. Since the fundamental group $\pi_1(N)$ acts isometrically on \tilde{N} by deck transformation, so given any F satisfying (6.3) and (6.4), we have that for all $\alpha \in \pi_1(N)$ the function $F_\alpha := (F \circ \alpha)$ also satisfies (6.3) and (6.4) (with y_0 replaced by $\alpha^{-1}(y_0)$). If N supports an n -obstruction ν_t , then its lift $\tilde{\nu}_t$ is also an n -obstruction on (\tilde{N}, \tilde{h}) . For any $\alpha \in \pi_1(M)$ we consider and application of the heat operator to $\omega_t = F_\alpha \circ \tilde{\nu}_t$ and obtain, using (6.3),

$$(\partial_t - \Delta)\omega_t = -c_0 |T\nu_t|^2.$$

Using the representation formula for the heat equation applied to ω yields, for $t_0 < 0$,

$$(4\pi)^{m/2} \omega_0(0) \leq |t_0|^{-m/2} \int_{\mathbb{R}^m} \omega_{t_0}(x) e^{-\frac{|x|^2}{4|t_0|}} dV - \int_{t_0}^0 \int_{\mathbb{R}^m} |T\nu_t|^2 |t|^{-m/2} e^{-\frac{|x|^2}{4|t|}} dV dt.$$

Since $w = F_\alpha(\tilde{\nu}) \geq 0$ by (6.4) then we have that

$$(6.5) \quad \int_{t_0}^0 \int_{\mathbb{R}^m} |T\nu_t|^2 |t|^{-m/2} e^{-\frac{|x|^2}{4|t|}} dV dt \leq c_0^{-1} |t_0|^{-m/2} \int_{\mathbb{R}^m} \omega_{t_0}(x) e^{-\frac{|x|^2}{4|t_0|}} dV dt.$$

There is a choice of $t_0 \in [-2, -1]$ and $\alpha \in \pi_1(N)$, verified in [11] pp.121-122 (we refer the reader to this verification) so that

$$(6.6) \quad |t_0|^{-m/2} \int_{\mathbb{R}^m} \omega_{t_0}(x) e^{-\frac{|x|^2}{2|t_0|}} dV < c_3,$$

where $c_3 := c_3(c_0, c_1, c_2, E_0) \in \mathbb{R}_{>0}$. If we take the combination of (6.5) and (6.6) and then apply a ‘layer cake’ decomposition of the time domain we obtain

$$\begin{aligned} \int_{t_0}^0 \int_{\mathbb{R}^m} |T\nu_t|^2 |t|^{-m/2} e^{-\frac{|x|^2}{4|t|}} dV dt &< \sum_{k=1}^{\infty} \int_{-4^{-k}}^{-4^{-(k+1)}} \int_{\mathbb{R}^m} |T\nu_t|^2 G_0 dV dt \\ &\leq c_0^{-1} c_3 \\ &= c_4 < +\infty. \end{aligned}$$

This implies that for a given $\delta > 0$ there is some index k_δ within the summation over which

$$I_{k_\delta} := \int_{-4^{-k_\delta}}^{-4^{-(k_\delta+1)}} \int_{\mathbb{R}^m} |T\nu_t|^2 G_0 dV dt \leq \delta^2.$$

We may choose $\delta \in \mathbb{R}_{>0}$ sufficiently small coming from Proposition 7.1.4 in [11] which yields that one has $|T\nu_0|(0) \leq c_5$ for some $c_5 = c_5(\delta, n, \tilde{N}, E_0)$.

Conversely, for every $\kappa \in \mathbb{R}_{>0}$ we can construct another n -obstruction map via rescaling with $\nu_t^\kappa(x) = \nu_{\kappa^2 t}(\kappa x)$. We can apply, for each one of these n -obstruction tensors, the same argument. Furthermore if we take a limit over $\kappa \rightarrow \infty$ their limit will also be an n -obstruction tensor. But then we have

$$\lim_{\kappa \rightarrow \infty} |T\nu_0^\kappa|(0) = \lim_{\kappa \rightarrow \infty} \kappa |T\nu_0|(0) = \infty.$$

So the limit cannot be an n -obstruction because (nO2) of the definition is violated. We conclude an n -obstruction tensor cannot exist on N , and the result follows. \square

With these preliminaries in place we can establish Theorem 1.4.

Proof of Theorem 1.4. We consider the solution f_t to harmonic map flow with initial condition $f : M \rightarrow \mathring{N}$. Let \overline{N} denote the doubling of N , i.e. $\overline{N} = N \sqcup_{\partial N} N$. Using cutoff functions in a small neighborhood of the boundary we can double the metric h as well. We can interpret the solution to the harmonic map flow with initial condition f as a flow into \overline{N} , and then we know by Proposition 6.8 that if the flow encounters a singularity then there exists an n -obstruction mapping into \overline{N} .

We further claim that for all $t \geq 0$, $f_t(M) \subset \mathring{N}$ by a strong maximum principle argument. Let F be the strictly convex boundary-defining function on $(N, \partial N, h)$, where $N = F^{-1}(-\infty, c]$ and $\partial N = F^{-1}(c)$. Note that there exists some $\epsilon > 0$ such that $F \circ f_0 \leq c - \epsilon$. Moreover, since N is compact, $F^{-1}(-\infty, c - \epsilon]$ is also compact, and hence the only way the flow can leave this set is by increasing F beyond the threshold of $c - \epsilon$.

So, let $\Phi_t = F \circ f_t$. A direct computation shows that

$$\frac{\partial}{\partial t} [\Phi_t] = \Delta \Phi_t - \nabla^2 F(Tf_\alpha, Tf_\alpha) \leq \Delta \Phi_t.$$

It follows that $\sup_M \Phi_t \leq \sup_M \Phi_0$ for all $t \geq 0$, and hence $f_t \subset F^{-1}(-\infty, c - \epsilon] \subset \mathring{N}$. Thus, the construction of Proposition 6.8 in fact generates an n -obstruction into \mathring{N} , which admits a strictly convex function. This is a contradiction as outlined in the proof of Theorem 6.9. Hence the solution exists for all time with a uniform estimate on the energy density. This implies subconvergence to a limiting harmonic map, which is necessarily a constant map by ([6]). \square

6.3. Conjectural framework for stable singularities into spheres. Let us now discuss the strategy of dimensional augmentation and how it relates to singularity formation for the harmonic map heat flow. Consider a harmonic map heat flow $f_t : M \rightarrow S^n$ which encounters a singularity at $T < \infty$. We can use a hemispherical embedding $S^n \subset S^{n+1}$ and think of the solution as a family $\tilde{f}_t : M \rightarrow S^{n+1}$.

We have already exhibited several ways in which the singularity encountered by \tilde{f} is unstable. In particular Corollary 5.9 shows that any soliton singularity model for \tilde{f} will be unstable or constant. Alternatively, one can use Theorem 1.4 to exhibit the instability. In particular, for any time $t < T$, one can perturb the map \tilde{f}_t into the upper x_{n+2} hemisphere via

$$\bar{f}_\epsilon = \left(\sqrt{1 - \epsilon^2} f^1, \dots, \sqrt{1 - \epsilon^2} f^{n+1}, \epsilon \right),$$

where we have used the explicit components of the given map f . The image lies entirely in a sublevel set of the strictly convex function x_{n+2} , and hence Theorem 1.4 guarantees the long time existence and convergence of the flow with initial condition \bar{f}_ϵ .

This gives an alternative way of moving past even a stable singularity of the flow mapping into spheres. In particular, one could choose a sequence of times $t_i \rightarrow T$ and a sequence $\epsilon_i \rightarrow 0$, and then consider the long time solutions to the flow with initial condition $\bar{f}_{\epsilon_i}(t_i)$, and consider the limit as i goes to infinity. A similar strategy was employed by Grayson and Altschuler [1] for the curve shortening flow, following a suggestion of Calabi as follows. Consider a solution $\gamma_t(s)$ to the curve shortening flow in the plane which has a singularity at $t = T < \infty$. Consider so called ramps over γ , that is embedded curves in \mathbb{R}^3 with $\frac{d\gamma_3}{ds} > 0$ and which project to parameterizations of γ . One then takes a sequence of ramps which approximate γ and for which the curve shortening flow for this sequence exists on a uniformly longer time interval, and then takes the approximation parameter to zero. Once the right estimates are obtained, the limiting flow exists for all time and

agrees with the evolution of γ away from singularities. Constructing a “flow past singularities” for the harmonic map heat flow may be approachable using similar methods.

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